Assignment 8

1. Suppose $u(x, y, t)$ is defined, for all $t \geq 0$, on the rectangle given by $0 \leq x \leq 2$, $0 \leq y \leq 3$, and $t \geq 0$. Suppose that for all $t \geq 0$,

$$\frac{\partial u}{\partial x}(0, y) = y^2, \qquad \frac{\partial u}{\partial x}(2, y) = 1, \qquad \frac{\partial u}{\partial y}(x, 0) = -7, \quad \text{and} \quad \frac{\partial u}{\partial y}(x, 3) = x.$$

a. If u satisfies the heat equation $\frac{\partial u}{\partial t} = \nabla^2 u$ within the rectangle, find

$$\frac{d}{dt} \int_0^3 \int_0^2 u \; dx \; dy.$$

b. Give a reason why there can not exist an equilibrium solution to the heat equation with the given boundary conditions. (An equilibrium solution is one which does not depend on t.)

2. Again suppose $u(x, y, t)$ is defined for $0 \leq x \leq 2$, $0 \leq y \leq 3$, and $t \geq 0$, but now suppose that for all $t \geq 0$,

$$\frac{\partial u}{\partial x}(0, y) = y^2, \qquad \frac{\partial u}{\partial x}(2, y) = 1, \qquad \frac{\partial u}{\partial y}(x, 0) = -2, \quad \text{and} \quad \frac{\partial u}{\partial y}(x, 3) = x.$$

a. If u satisfies the heat equation $\frac{\partial u}{\partial t} = \nabla^2 u$ within the rectangle, find

$$\frac{d}{dt} \int_0^3 \int_0^2 u \; dx \; dy.$$

b. Do you think there exists an equilibrium solution to the heat equation with the given boundary conditions? Why? You do not have to find an equilibrium solution, but write a few words explaining how you might go about finding one.

3. Suppose $u(r, \theta, t)$ is defined, for all $t \geq 0$, on the circle given by $0 \leq r \leq 2$ and $0 \leq \theta \leq 2\pi$. Suppose that for all $t \geq 0$,

$$\frac{\partial u}{\partial r}(2, \theta) = \cos\theta.$$

a. If u satisfies the heat equation $\frac{\partial u}{\partial t} = \nabla^2 u$ within the circle, find

$$\frac{d}{dt} \int_0^{2\pi} \int_0^2 u \; r \; dr \; d\theta.$$

b. Do you think there exists an equilibrium solution to the heat equation with the given boundary conditions? Why? See if you can guess an equilibrium solution. (Hint: try very simple functions of x and/or y.)