Solutions to Problems on Assignment 9

4.4.2(c) The PDE we have to solve is

$$\rho_0 \frac{\partial^2 u}{\partial t^2} = T_0 \frac{\partial^2 u}{\partial x^2} + \alpha u,$$

where ρ_0, T_0, α are constants, and $\alpha < 0$.

First find separated solutions which satisfy the boundary conditions: putting $u(x,t) = \phi(x)h(t)$, into the PDE and dividing by $\rho_0 \phi h$, we get

$$\frac{\frac{d^2h}{dt^2}}{h} = \frac{T_0}{\rho_0}\frac{\frac{d^2\phi}{dx^2}}{\phi} + \frac{\alpha}{\rho_0}.$$

Since the left side is a function of t alone and the right side is a function of x alone, both sides must equal a constant. Let's call the constant $-\gamma$ (so the ODE for h will have a familiar form). Then we get the ODEs

$$\frac{d^2h}{dt^2} = -\gamma h$$

and

$$\frac{d^2\phi}{dx^2} = \left[\left(-\gamma - \frac{\alpha}{\rho_0}\right)\frac{\rho_0}{T_0}\right]\phi.$$

The latter equation is just of the form $\frac{d^2\phi}{dx^2} = (\text{constant})\phi$, which we already know how to solve, except we usually call the constant $-\lambda$. So let's call the constant $-\lambda$ here too; i.e.,

$$-\lambda = \left[\left(-\gamma - \frac{\alpha}{\rho_0}\right)\frac{\rho_0}{T_0}\right].$$

As we know, the solutions for ϕ (with boundary conditions $\phi(0) = \phi(L) = 0$ are given by $\phi(x) = \sin(\sqrt{\lambda}x)$ where $\lambda = (n\pi/L)^2$, for $n = 1, 2, 3, \ldots$. Putting these values of λ into the preceding equation and solving for γ, we get

$$\gamma = \frac{(n\pi/L)^2 T_0 - \alpha}{\rho_0}.$$

Now solving the ODE for $h(t)$ gives

$$h(t) = A\cos(\sqrt{\gamma}t) + B\sin(\sqrt{\gamma}t),$$

so the separated solutions are of the form

$$(A\cos(\sqrt{\gamma}t) + \sin B(\sqrt{\gamma}t))\sin(\sqrt{\lambda}x),$$

where λ and γ are given by the above formulas.

Finally we have to find a superposition of the separated solutions which satisfies the correct initial conditions. Put

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\cos(\sqrt{\gamma}_n t) + B_n\sin(\sqrt{\gamma}_n t)\right)\sin((n\pi/L)x),$$

where

$$\gamma_n = \frac{(n\pi/L)^2 T_0 - \alpha}{\rho_0}.$$

(I've attached a subscript n to the letter γ to emphasize that it depends on n. Notice that what I am calling γ_n here is called λ_n in the answer to this problem in the back of the book.)

Since $u(x,0) = 0$, then

$$0 = \sum_{n=1}^{\infty} A_n \sin((n\pi/L)x),$$

which tells us that $A_n = 0$ for all n. Next, differentiating u with respect to t gives

$$\frac{\partial u}{\partial t}(x,t) = \sum_{n=1}^{\infty} B_n \sqrt{\gamma_n} \cos(\sqrt{\gamma_n} t) \sin((n\pi/L)x),$$

and putting $\frac{\partial u}{\partial t}(x,0) = f(x)$ gives

$$f(x) = \sum_{n=1}^{\infty} B_n \sqrt{\gamma_n} \sin((n\pi/L)x),$$

which tells us that

$$B_n = \frac{2}{\sqrt{\gamma_n} L} \int_0^L f(w) \sin((n\pi/L)w)\ dw.$$

Therefore the solution to the problem is

$$u(x,t) = \sum_{n=1}^{\infty} B_n \sin(\sqrt{\gamma_n} t) \sin((n\pi/L)x),$$

where B_n and γ_n are given by the above formulas.

As discussed in class, the number $\sqrt{\gamma_n}$ is what is usually called the *circular frequency* of this oscillation. The term *frequency* would usually be reserved for the number $\sqrt{\gamma_n}/(2\pi)$.

4.4.3(b) As in the preceding problem we put $u(x,t) = \phi(x)h(t)$ into the PDE and divide by $\rho_0 \phi h$ to obtain

$$\frac{\frac{d^2h}{dt^2}}{h} = \frac{T_0}{\rho_0} \frac{\frac{d^2\phi}{dx^2}}{\phi} - \frac{\beta}{\rho_0} \frac{\frac{dh}{dt}}{h}.$$

which we can rewrite as

$$\frac{\rho_0}{T_0} \left[\frac{\frac{d^2h}{dt^2}}{h} + \frac{\beta}{\rho_0} \frac{\frac{dh}{dt}}{h} \right] = \frac{\frac{d^2\phi}{dx^2}}{\phi}.$$

Setting both sides equal to the constant $-\lambda$, we get for ϕ the usual ODE $\frac{d^2\phi}{dx^2} = -\lambda\phi$ and boundary conditions $\phi(0) = \phi(L) = 0$. Thus $\lambda = -(n\pi/L)^2$ and $\phi(x) = \sin((n\pi/L)x)$. For $h(t)$ we get the ODE

$$\frac{d^2h}{dt^2} + \frac{\beta}{\rho_0} \frac{dh}{dt} = -\left(\frac{\lambda T_0}{\rho_0} \right) h.$$

The characteristic equation for this ODE is

$$r^2 + \left(\frac{\beta}{\rho_0} \right) r + \left(\frac{\lambda T_0}{\rho_0} \right) = 0,$$

and using the quadratic equation to solve it gives

$$r = \frac{\left(\frac{-\beta}{\rho_0} \right) \pm \sqrt{\left(\frac{\beta}{\rho_0} \right)^2 - 4 \left(\frac{\lambda T_0}{\rho_0} \right)}}{2} = \frac{-\beta}{2\rho_0} \pm \sqrt{\frac{\beta^2}{4\rho_0^2} - \left(\frac{\lambda T_0}{\rho_0} \right)} = \frac{-\beta}{2\rho_0} \pm \sqrt{\frac{\beta^2}{4\rho_0^2} - \frac{T_0}{\rho_0} \left(\frac{n\pi}{L} \right)^2}.$$

Since the problem says we can assume that $\beta^2 < 4\phi^2 \rho_0 T_0 / L^2$, then in the last equation the quantity under the radical is negative, and so can be written as $-w_n^2$, where w_n is given by

(1) $$w_n = \sqrt{\frac{T_0}{\rho_0} \left(\frac{n\pi}{L} \right)^2 - \frac{\beta^2}{4\rho_0^2}}.$$

We then have

$$r = -(\beta/2\rho_0) \pm i w_n.$$

The general solution of the ODE for h is now given by

$$\begin{aligned} h(t) &= P e^{(-(\beta/2\rho_0)+iw_n)t} + Q e^{(-(\beta/2\rho_0)-iw_n)t} \\ &= e^{-\beta t/2\rho_0}(P e^{iw_n t} + Q e^{-iw_n t}) \\ &= e^{-\beta t/2\rho_0}(A\cos(w_n t) + B\sin(w_n t)), \end{aligned}$$

where A and B are arbitrary constants.

So, separated solutions of the PDE and boundary conditions are of the form

$$e^{-\beta t/2\rho_0}\left[A\cos(w_n t) + B\sin(w_n t)\right]\sin(n\pi x/L).$$

Next, we look for a superposition

$$u(x,t) = \sum_{n=1}^{\infty} e^{-\beta t/2\rho_0}\left[A_n\cos(w_n t) + B_n\sin(w_n t)\right]\sin(n\pi x/L) \tag{2}$$

which satisfies the given initial conditions. Putting $t = 0$ and using $u(x,0) = f(x)$, we find that

$$f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x/L),$$

so

$$A_n = \frac{2}{L}\int_0^L f(w)\sin(n\pi w/L)\ dw. \tag{3}$$

Also, taking the derivative of $u(x,t)$ with respect to t gives

$$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left\{-\frac{\beta}{2\rho_0}e^{-\beta t/2\rho_0}\left[A_n\cos(w_n t) + B_n\sin(w_n t)\right] + e^{-\beta t/2\rho_0}\left[-A_n w_n\sin(w_n t) + B_n w_n\cos(w_n t)\right]\right\}\sin(n\pi x/L),$$

and putting $t = 0$ gives

$$g(x) = \sum_{n=1}^{\infty}\left[-\left(\frac{\beta}{2\rho_0}\right)A_n + B_n w_n\right]\sin(n\pi x/L).$$

Therefore

$$\left[-\left(\frac{\beta}{2\rho_0}\right)A_n + B_n w_n\right] = \frac{2}{L}\int_0^L g(w)\sin(n\pi w/L)\ dw,$$

and solving for B_n gives

$$B_n = \frac{2}{L w_n}\int_0^L g(w)\sin(n\pi w/L)\ dw + \frac{\beta A_n}{2\rho_0 w_n}. \tag{4}$$

The solution is therefore given by the series in equation (2), where w_n is given by equation (1), A_n is given by equation (3), and B_n is given by equation (4).