1. (problem #7, page 312 of Royden and Fitzpatrick) Suppose X is a vector space over the real numbers, and is a normed space, with norm $\|\cdot\|$. We say that the norm is induced by an inner product if (i) there exists an inner product defined on X, that is, for all u, v in X there is defined a real number $\langle u, v\rangle$ with the properties in the definition on page 309; and (ii) for all $u \in X$, $\langle u, u\rangle = \|u\|^2$.

We already know (see page 310) that if the norm on X is induced by an inner product, then for all u and v in X we have

$$\|u - v\|^2 + \|u + v\|^2 = 2\|u\|^2 + 2\|v\|^2.$$

Prove that, conversely, if we are given that the above identity holds for all u and v in X, then the norm on X must be induced by an inner product. Hint: you can define the inner product as

$$\langle u, v\rangle := \frac{1}{4}\left(\|u + v\|^2 - \|u - v\|^2\right).$$

2. Suppose $f \in L^1(\mathbf{R})$. For every Lebesgue measurable subset E of $\mathbf{R}$, define

$$\phi(E) = \int_E f$$

Show that:

(i) This function ϕ is an additive set function on $\mathbf{R}$.

(ii) For every measurable set E, we have $\overline{V}(E) = \int_E f^+$, $\underline{V}(E) = \int_E f^-$, and $V(E) = \int |f|$, where $\overline{V}$, $\underline{V}$, and V are as defined in class.

(iii) If we set $P = \{x : f(x) \geq 0\}$ and $N = \{x : f(x) < 0\}$, then P and N form a Hahn decomposition for ϕ: that is, $P \cup N = \mathbf{R}$, $P \cap N = \emptyset$, and P is a positive set and N is a negative set for ϕ.

3. Do problem 19 on page 372 of Royden and Fitzpatrick. You can use any result from Section 18.2 of Royden and Fitzpatrick or from chapter 10 of Wheeden and Zygmund.