Theoretical underpinnings of differential calculus

Intermediate Value Theorem: Suppose $f(x)$ is continuous at every point of a closed interval $[a, b]$, and $f(a)$ is positive and $f(b)$ is negative. Then there is some point in between a and b where f takes the value zero; i.e., there exists some c in $[a, b]$ such that $f(c) = 0$.

Extreme Value Theorem: Suppose $f(x)$ is continuous at every point of a closed interval $[a, b]$. Then there is some point in between a and b where f takes its absolute maximum value over all of $[a, b]$. That is, there exists some c in $[a, b]$ such that $f(c)$ is greater than or equal to $f(x)$ for all other values of x in $[a, b]$. Similarly, there is some point in between a and b where f takes its absolute minimum value over all of $[a, b]$. That is, there exists some d in $[a, b]$ such that $f(d)$ is less than or equal to $f(x)$ for all other values of x in $[a, b]$.

Theorem (called "Fermat's Theorem" in our text): Suppose f has a local extremum (either a local maximum or a local minimum) at some point. Then, if f is differentiable at that point, the derivative of f must be zero there. That is, if f is differentiable at c and has either a local maximum or a local minimum at c, then $f'(c)$ has to equal zero.

Note: The converse to Fermat's Theorem is not true! It is possible for the derivative of f to equal zero at a point without there being a local extremum of f at that point.

Mean Value Theorem: Suppose $f(x)$ is continuous at every point of a closed interval $[a, b]$, and differentiable at every point of the interior interval (a, b). Then there is some point in between a and b where the slope of the graph of f is equal to the slope of the line connecting the points on the graph of f where $x = a$ and $x = b$. That is, there exists some c in $[a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Rolle's Theorem: (This is a special case of the Mean Value Theorem.) Suppose $f(x)$ is continuous at every point of a closed interval $[a, b]$, and differentiable at every point of the interior interval (a, b). Suppose $f(a) = 0$ and $f(b) = 0$. There there is some point in between a and b where the derivative of f is zero. That is, there exists some c in $[a, b]$ such that $f'(c) = 0$.