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Math 4433
Exam 1

Name: key

1. *(15 points)*
a. Give the definition of supremum of a set.

Suppose $S \subseteq \mathbb{R}$ and $u \in \mathbb{R}$. We say u is a supremum of S if:

[5] (i) for all $s \in S$, $s \le u$; and

[5] (ii) if $v \in \mathbb{R}$ and $s \le v$ for all $s \in S$, then $u \le v$.

[5] **b.** State the completeness property of **R**.

~~If~~ Every nonempty subset of $\mathbb{R}$ which is bounded above has a supremum.

2. *(20 points)*
a. Give the definition of limit of a sequence.

[5] Suppose (x_n) is a sequence and $x \in \mathbb{R}$. We say $\lim(x_n) = x$ if, for every (1) $\varepsilon > 0$, there exists (1) $K \in \mathbb{N}$ such that for all $n \ge K$ (1), $|x_n - x| < \varepsilon$ (2).

[15] **b.** Prove that if (x_n) converges to x and (y_n) converges to y, then $(x_n + y_n)$ converges to $x + y$.

Let $\varepsilon > 0$ be given. (1) Choose $K_1 \in \mathbb{N}$ (1) s.t. if $n \ge K_1$, then $|x_n - x| < \frac{\varepsilon}{2}$ (2); and choose $K_2 \in \mathbb{N}$ (1) s.t. if $n \ge K_2$, then $|y_n - y| < \frac{\varepsilon}{2}$ (2).

Let $K = \max\{K_1, K_2\}$ (2). If $n \ge K$ (1), then

$$|(x_n+y_n)-(x+y)| \overset{(1)}{=} |(x_n-x)+(y_n-y)| \overset{(1)}{\le} |x_n-x| + |y_n-y| \overset{(2)}{}$$

$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \text{ (1)}$$

3. *(15 points)* Let $S = \{1 - \frac{1}{n} : n \in \mathbf{N}\}$.

a. Suppose $v \in \mathbf{R}$ and $v < 1$. Show that v is not an upper bound of S. 2

Since $v < 1$, then $1 - v > 0$, By the Archimedean principle, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{1-v}$. 4

3 Hence $1 - v > \frac{1}{n}$, so $1 - \frac{1}{n} > v$. This shows there is an element of S which is greater than v, so v is not an upper bound of S.

b. Find, with proof, the supremum of S. 2

We claim that the supremum of S is 1.

Since $1 - \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$, then 1 is an upper bound of S. 2

Also, by part a., every upper bound v of S satisfies $v \geq 1$. 2

So, by def of supremum, $1 = \sup S$.

4. *(15 points)* Suppose $x_n \neq 0$ for all $n \in \mathbf{N}$, $\lim(x_n) = x$, and $x \neq 0$. Show that the sequence $(\frac{1}{x_n})$ is bounded. (You may use any theorem from class, but remember to indicate when you are using one.)

Since (x_n) converges to x, and $x \neq 0$, it follows

[8] from a theorem proved in class that $(\frac{1}{x_n})$ converges.

[7] Since $(\frac{1}{x_n})$ converges, it follows from another theorem proved in class that $(\frac{1}{x_n})$ is bounded.

Alternate solution: First suppose $x > 0$. 2 Choose $\varepsilon > 0$ so that $x - \varepsilon > 0$ 3 (for example, choose $\varepsilon = \frac{x}{2}$). Since $\lim(x_n) = x$, there exists $K \in \mathbb{N}$ s.t. if $n \geq K$ then $|x_n - x| < \varepsilon$. So $n \geq K \Rightarrow -\varepsilon + x < x_n < \varepsilon + x$. 5

Since $0 < -\varepsilon + x$, then $0 < x_n$ also, so $-\varepsilon + x < x_n$ implies $0 < \frac{1}{x_n} < \frac{1}{-\varepsilon + x}$. 2

Let $M = \max\{|\frac{1}{x_1}|, |\frac{1}{x_2}|, \ldots, |\frac{1}{x_{K-1}}|, \frac{1}{-\varepsilon + x}\}$; then $|\frac{1}{x_n}| \leq M$ for all $n \in \mathbb{N}$. 3

Finally, if $x < 0$, then apply the preceding argument to $(-x_n)$ to get that $(-\frac{1}{x_n})$ is bounded. It follows that $(\frac{1}{x_n})$ is too.

5. *(20 points)* Find, with proof, the limit of the sequence $\left(\frac{(-1)^n n}{n^2+1}\right)$.

For all $n \in \mathbb{N}$ we have 5

$$\left|\frac{(-1)^n n}{n^2+1}\right| \le \frac{n}{n^2+1} \le \frac{n}{n^2} = \frac{1}{n}.$$

So, for all $n \in \mathbb{N}$, 5

$$-\frac{1}{n} \le \frac{(-1)^n n}{n^2+1} \le \frac{1}{n}.$$

From class we know that $\lim\left(\frac{1}{n}\right) = 0$, and hence 5

also $\lim\left(-\frac{1}{n}\right) = \lim(-1)\left(\frac{1}{n}\right) = (-1)\cdot 0 = 0$.

So by the Squeeze Theorem, $\lim\left(\frac{(-1)^n n}{n^2+1}\right) = 0$. 5

Alternate solution: Let $\varepsilon > 0$ be given. 2 Choose $K \in \mathbb{N}$ s.t. $K > \frac{1}{\varepsilon}$. 3

If $n \ge K$ then 2 $\left|\frac{(-1)^n n}{n^2+1} - 0\right| = \frac{n}{n^2+1} \le \frac{n}{n^2} = \frac{1}{n} \le \frac{1}{K} < \varepsilon$. 3 2

So $\lim\left(\frac{(-1)^n n}{n^2+1}\right) = 0$. 5

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6. *(15 points)* Find $\lim\left(\frac{1}{(1+\frac{1}{n})^{n^2}}\right)$. (Hint: use Bernoulli's inequality.)

By Bernoulli's Inequality, for all $n \in \mathbb{N}$, 3

$$\left(1+\frac{1}{n}\right)^{n^2} \ge 1 + n^2 \cdot \frac{1}{n} = 1 + n \ge n.$$ 2

So $\frac{1}{(1+\frac{1}{n})^{n^2}} \le \frac{1}{n}$. 5 But $\lim\left(\frac{1}{n}\right) = 0$. 2

Since $\frac{1}{(1+\frac{1}{n})^{n^2}} \ge 0$ and $\lim(0) = 0$, 3

it follows from the Squeeze Theorem that $\lim\left(\frac{1}{(1+\frac{1}{n})^{n^2}}\right) = 0$.