Math 4443/5443
Exam 1

Instructions: Do problems 1, 2 and any five of the six remaining problems (for a total of 100 possible points).

1. *(20 points)*
a. Give the definition of "Riemann integral of a function on $[a,b]$".
b. Prove that a function f can have no more than one Riemann integral on $[a,b]$.

2. *(20 points)* Prove that if f is differentiable on $[a,b]$ and $f'(x) = 0$ for all $x \in [a,b]$, then f is constant on $[a,b]$.

3. *(12 points)* Prove that if f and g are differentiable on $\mathbf{R}$, $f(0) = g(0) = 0$, and $f'(x) \leq g'(x)$ for all $x > 0$, then $f(x) \leq g(x)$ for all $x > 0$.

4. *(12 points)* Prove that $e^x - e^{-x} \geq 2x$ for all $x > 0$ (you may use problem **3**, even if you haven't proved it).

5. *(12 points)* Suppose $f''(x)$ exists for all $x \in \mathbf{R}$, and $f''(x) > 0$ for all $x \in \mathbf{R}$. Suppose also that $f(0) = f'(0) = 0$. Prove that $f(x) > 0$ for all $x \in \mathbf{R}$. (The easiest way to do this is to use Taylor's theorem.)

6. *(12 points)* Prove that if $f \in \mathcal{R}[a,b]$, then $\left|\int_a^b f \, dx\right| \leq \int_a^b |f| \, dx$.

7. *(12 points)* Either prove that the following statement is true, or give a counterexample to show that it is false:

"If $f \in \mathcal{R}[a,b]$ and $\int_a^b f = 0$, then $f(x) = 0$ for all $x \in [a,b]$."

8. *(12 points)* Suppose $f : [a,b] \to \mathbf{R}$, and suppose there exist sequences of tagged partitions $\{\dot{\mathcal{P}}_n\}$ and $\{\dot{\mathcal{Q}}_n\}$ such that $\|\dot{\mathcal{P}}_n\| < 1/n$ and $\|\dot{\mathcal{Q}}_n\| < 1/n$ for all n, and

$$\left|S(f, \dot{\mathcal{P}}_n) - S(f, \dot{\mathcal{Q}}_n)\right| \geq 1$$

for all n. Show that $f \notin \mathcal{R}[a,b]$. (You may use any theorem from class.)