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EXAM 2 -003 **Name** key
Math 3333
3-15-07

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(12 points)* Prove that $W = \left\{ \begin{bmatrix} s \\ t \\ 3t+2s \end{bmatrix} : s,t \in \mathbf{R} \right\}$ is a subspace of $\mathbf{R}^3$.

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Suppose $\vec{u} = \begin{bmatrix} s_1 \\ t_1 \\ 3t_1+2s_1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} s_2 \\ t_2 \\ 3t_2+2s_2 \end{bmatrix}$ are in W. Then $\vec{u}+\vec{v} = \begin{bmatrix} s_1+s_2 \\ t_1+t_2 \\ 3t_1+2s_1+3t_2+2s_2 \end{bmatrix}$

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$= \begin{bmatrix} s_1+s_2 \\ t_1+t_2 \\ 3(t_1+t_2)+2(s_1+s_2) \end{bmatrix}$, which is in W because it is of the form $\begin{bmatrix} s \\ t \\ 3t+2s \end{bmatrix}$.

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Also if $c \in \mathbb{R}$ then $c\vec{u} = \begin{bmatrix} cs_1 \\ ct_1 \\ c(3t_1+2s_1) \end{bmatrix} = \begin{bmatrix} cs_1 \\ ct_1 \\ 3(ct_1)+2(cs_1) \end{bmatrix}$ is in W. So W is closed under addition and scalar multiplication, and hence is a subspace.

2. *(16 points)* Let $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 12 \end{bmatrix} \right\}$.

12] a) Decide whether $\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$ is in W. Show all work.

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We have to check whether $x_1\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2\begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + x_3\begin{bmatrix} 2 \\ 1 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$ has a solution.

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Solving, we get $\left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 1 & 3 & 1 & 5 \\ 2 & 1 & 12 & 10 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & -3 & 8 & 6 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 5 & 15 \end{array}\right] \to$

$\to \left[\begin{array}{ccc|c} 1 & 2 & 2 & 2 \\ 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$. This gives the equations $\begin{aligned} x_1+2x_2+2x_3 &= 2 \\ x_2 - x_3 &= 3 \\ x_3 &= 3 \end{aligned}$, or $x_3 = 3$,

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$x_2 = 3 + x_3 = 6$, and $x_1 = 2 - 2x_2 - 2x_3 = 2 - 12 - 6 = -16$. So a solution does exist $(x_1 = -16,\ x_2 = 6,\ x_3 = 3)$, so $\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$ is in W.

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b) Does W equal $\mathbf{R}^3$? Explain your answer.

4] Yes. One possible explanation: replacing $\begin{bmatrix} 2 \\ 5 \\ 10 \end{bmatrix}$ by $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in part a) gives

$\left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 1 & 3 & 1 & b \\ 2 & 1 & 12 & c \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 0 & 1 & -1 & b-a \\ 0 & -3 & 8 & c-2a \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 5 & (c-2a)+3(b-a) \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 2 & a \\ 0 & 1 & -1 & b-a \\ 0 & 0 & 1 & \frac{1}{5}(c-2a)+\frac{3}{5}(b-a) \end{array}\right]$,

which has a solution no matter what a, b, and c are. So every vector $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ in $\mathbb{R}^3$ is in W.

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3. *(10 points)* Let S be the basis of $\mathbf{R}^3$ given by $S = \left\{ \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} \right\}$. Let $\mathbf{v} = \begin{bmatrix} 8 \\ 9 \\ 12 \end{bmatrix}$. Find the coordinate vector $[\mathbf{v}]_S$.

We have $[v]_S = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ where $a_1 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 0 \\ 3 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 8 \\ 9 \\ 12 \end{bmatrix}$. 2

Solving for a_1, a_2, a_3: $\left[\begin{array}{ccc|c} 2 & 0 & 0 & 8 \\ 0 & 3 & 0 & 9 \\ 0 & 0 & 4 & 12 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & 3 \end{array}\right]$, so
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$a_1 = 4$
$a_2 = 3$, and $[v]_S = \begin{bmatrix} 4 \\ 3 \\ 3 \end{bmatrix}$.
$a_3 = 3$
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4. *(8 points)* Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in a vector space V, and $2\mathbf{v}_1 + 5\mathbf{v}_2 + 6\mathbf{v}_3 = 11\mathbf{v}_1 + 7\mathbf{v}_3$. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set.

If $2\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_3 = 11\vec{v}_1 + 7\vec{v}_3$,

then $2\vec{v}_1 - 11\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_3 - 7\vec{v}_3 = \vec{0}$, 2

so $(2-11)\vec{v}_1 + 5\vec{v}_2 + (6-7)\vec{v}_3 = \vec{0}$, 2

so $(-9)\vec{v}_1 + 5\vec{v}_2 + (-1)\vec{v}_3 = \vec{0}$, so the equation $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$ has a nontrivial solution 4 $(a_1 = -9, a_2 = 5, a_3 = -1)$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

5. *(10 points)* Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in a vector space V, and $\mathbf{v}_3$ is in span$\{\mathbf{v}_1, \mathbf{v}_2\}$. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set.

~~Suppose~~ By assumption, $\vec{v}_3$ is a linear combination of $\vec{v}_1$ and $\vec{v}_2$, so $\vec{v}_3 = c_1\vec{v}_1 + c_2\vec{v}_2$ for some numbers $c_1, c_2 \in \mathbb{R}$. 4 So $c_1\vec{v}_1 + c_2\vec{v}_2 + (-1)\vec{v}_3 = \vec{0}$. So the equation $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$ 4 has a solution $(a_1 = c_1, a_2 = c_2, a_3 = -1)$, and this solution is nontrivial because $a_3 = -1 \neq 0$. 2 Therefore $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a linearly dependent set.

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6. *(44 points)* Let $A = \begin{bmatrix} 2 & 4 & 8 & 6 \\ 0 & 1 & 1 & 7 \\ 4 & 5 & 13 & 3 \end{bmatrix}$. Find the following, giving complete explanations for your answer. (You can use whichever methods you prefer.)

a) A basis for the row space of A

[12] We put A in row echelon form: $\begin{bmatrix} 2 & 4 & 8 & 6 \\ 0 & 1 & 1 & 7 \\ 4 & 5 & 13 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 1 & 7 \\ 0 & -3 & -3 & 9 \end{bmatrix} \to$ (4)

$\to \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 4 & 3 \\ 0 & 1 & 1 & 7 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (4) So a basis for the row space of A

is formed by the non-zero rows: $\{[1\ 2\ 4\ 3], [0\ 1\ 1\ 7], [0\ 0\ 0\ 1]\}$. (4)

b) A basis for the column space of A

10] (8) In the row-echelon form of A (found in part a)), only columns 1, 2, and 4 have initial 1's, so a basis for the column space of A is formed by columns 1, 2, and 4 of A:

(2) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 6 \\ 7 \\ 3 \end{bmatrix} \right\}$.

c) A basis for the solution space of A

14] Solutions of $A\vec{x} = \vec{0}$ are found by putting the augmented matrix in row echelon form: From part a) we see that the result is

$\left[\begin{array}{cccc|c} 1 & 2 & 4 & 3 & 0 \\ 0 & 1 & 1 & 7 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$ (2), giving equations $x_4 = 0$; (2)

$x_3 = t$ (free variable); (2)

$x_2 + x_3 + 7x_4 = 0 \Rightarrow x_2 + x_3 = 0 \Rightarrow x_2 = -t$; (2)

$x_1 + 2x_2 + 4x_3 + 3x_4 = 0 \Rightarrow x_1 = -2(-t) - 4t = -2t$, (2)

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2t \\ -t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix}$ (2), and a basis for the solution space is $\left\{ \begin{bmatrix} -2 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$. (2)

] d) Rank of A $= \dim(\text{row space}) = \dim(\text{column space}) = 3$

] e) Nullity of A $= \dim(\text{solution space}) = 1$.

N.