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EXAM 2 -004
Math 3333
3-15-07

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(12 points)* Prove that $W = \left\{ \begin{bmatrix} s \\ 4t - s \\ t \end{bmatrix} : s, t \in \mathbf{R} \right\}$ is a subspace of $\mathbf{R}^3$.

(2) (2) (2)

Let $\vec{u} = \begin{bmatrix} s_1 \\ 4t_1 - s_1 \\ t_1 \end{bmatrix} \in W$ and $\vec{v} = \begin{bmatrix} s_2 \\ 4t_2 - s_2 \\ t_2 \end{bmatrix} \in W$. Then $\vec{u} + \vec{v} = \begin{bmatrix} s_1 + s_2 \\ 4t_1 - s_1 + 4t_2 - s_2 \\ t_1 + t_2 \end{bmatrix} =$

(2)

$= \begin{bmatrix} s_1 + s_2 \\ 4(t_1 + t_2) - (s_1 + s_2) \\ t_1 + t_2 \end{bmatrix}$, which is in W because it is of the form $\begin{bmatrix} s \\ 4t - s \\ t \end{bmatrix}$,

(2) (2)

and $c\vec{u} = \begin{bmatrix} cs_1 \\ c(4t_1 - s_1) \\ ct_1 \end{bmatrix} = \begin{bmatrix} cs_1 \\ 4(ct_1) - (cs_1) \\ ct_1 \end{bmatrix}$, which is also in W. So W is closed under addition and scalar multiplication and is hence a subspace.

2. *(16 points)* Let $W = \operatorname{span}\left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 12 \end{bmatrix} \right\}$.

a) Decide whether $\begin{bmatrix} 2 \\ 5 \\ 11 \end{bmatrix}$ is in W. Show all work.

[12] (2)

We have to see whether $x_1 \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 3 \\ 1 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 1 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 11 \end{bmatrix}$ has a solution.

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So we solve $\left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 1 & 3 & 1 & 5 \\ 2 & 1 & 12 & 11 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & -3 & 6 & 7 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & 3 & 2 \\ 0 & 1 & -2 & 3 \\ 0 & 0 & 0 & 16 \end{array}\right]$

The last equation is $0 \cdot x_1 + 0 x_2 + 0 \cdot x_3 = 16$, which is impossible, so there is no solution. Hence $\begin{bmatrix} 2 \\ 5 \\ 11 \end{bmatrix}$ is not in W.

(2) (2)

[4] b) Does W equal $\mathbf{R}^3$? Explain your answer.

No, because W does not contain $\begin{bmatrix} 2 \\ 5 \\ 11 \end{bmatrix}$, which is in $\mathbb{R}^3$.

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3. *(10 points)* Let S be the basis of $\mathbf{R}^3$ given by $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} \right\}$. Let $\mathbf{v} = \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix}$. Find the coordinate vector $[\mathbf{v}]_S$.

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We have $[v]_S = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}$ where $a_1 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + a_2 \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + a_3 \begin{bmatrix} 0 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 9 \end{bmatrix}$.

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So we solve: $\left[\begin{array}{ccc|c} 0 & 2 & 0 & 4 \\ 1 & 0 & 0 & 7 \\ 0 & 0 & 3 & 9 \end{array}\right] \to \left[\begin{array}{ccc|c} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 7 \\ 0 & 0 & 1 & 3 \end{array}\right] \to \begin{matrix} a_2 = 2 \\ a_1 = 7 \\ a_3 = 3 \end{matrix} \to \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$.

So $[v]_S = \begin{bmatrix} 7 \\ 2 \\ 3 \end{bmatrix}$. 2

4. *(8 points)* Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in a vector space V, and $2\mathbf{v}_1 + 5\mathbf{v}_2 + 6\mathbf{v}_3 = 11\mathbf{v}_1 + 7\mathbf{v}_3$. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly dependent set.

If $2\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_3 = 11\vec{v}_1 + 7\vec{v}_3$,

then $2\vec{v}_1 - 11\vec{v}_1 + 5\vec{v}_2 + 6\vec{v}_3 - 7\vec{v}_3 = \vec{0}$, 2

so $(2-11)\vec{v}_1 + 5\vec{v}_2 + (6-7)\vec{v}_3 = \vec{0}$, 2

so $-9\vec{v}_1 + 5\vec{v}_2 + (-1)\vec{v}_3 = \vec{0}$, so the equation $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$ has a nontrivial solution 4 $(a_1 = -9, a_2 = 5, a_3 = -1)$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly dependent.

5. *(10 points)* Suppose $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ are vectors in a vector space V, and $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a linearly independent set. Show that $T = \{2\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is also a linearly independent set.

Suppose $a_1(2\vec{v}_1) + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$. ~~2~~ 3

Then $(2a_1)\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$.

Since $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are independent, this can only happen if $2a_1 = 0$, $a_2 = 0$, and $a_3 = 0$. 3

But then $2a_1 = 0$ implies $a_1 = 0$. 1

So the only solution of $a_1(2\vec{v}_1) + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$ is the trivial solution. So $\{2\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is independent.

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6. *(44 points)* Let $A = \begin{bmatrix} 2 & 6 & -2 & 10 \\ 0 & 1 & 2 & 4 \\ 1 & 6 & 5 & 20 \end{bmatrix}$. Find the following, giving complete explanations for your answer. (You can use whichever methods you prefer.)

a) A basis for the row space of A

[12] We put A in row echelon form: $\begin{bmatrix} 2 & 6 & -2 & 10 \\ 0 & 1 & 2 & 4 \\ 1 & 6 & 5 & 20 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 3 & 6 & 15 \end{bmatrix} \to$ (4)

$\to \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & -1 & 5 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (4) So a basis for the row space of A is formed by the non-zero rows: (4)

$\{[1 \ 3 \ -1 \ 5], [0 \ 1 \ 2 \ 4], [0 \ 0 \ 0 \ 1]\}$.

b) A basis for the column space of A

[10] (8) In the row echelon form of A that we found in a), we see that only columns 1, 2, and 4 have initial 1's; so a basis for the column space of A ~~contains~~ columns 1, 2, and 4 of A: (2)

$\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 6 \\ 1 \\ 6 \end{bmatrix}, \begin{bmatrix} 10 \\ 4 \\ 20 \end{bmatrix} \right\}$.

c) A basis for the solution space of A

[14] Solutions of $A\vec{x} = \vec{0}$ are found by putting $\left[A \,\middle|\, \begin{smallmatrix} 0 \\ 0 \\ 0 \end{smallmatrix} \right]$ in row echelon form; from part a) we see that the result is (2)

$\left[\begin{array}{cccc|c} 1 & 3 & -1 & 5 & 0 \\ 0 & 1 & 2 & 4 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{array}\right]$, giving

$x_4 = 0$ (2)

$x_3 = t$ (free variable) (2)

$x_2 + 2x_3 + 4x_4 = 0 \Rightarrow x_2 + 2t = 0 \Rightarrow x_2 = -2t$ (2)

$x_1 + 3x_2 - x_3 + 5x_4 = 0 \Rightarrow x_1 = -3(-2t) + t = 7t$ (2)

So $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 7t \\ -2t \\ t \\ 0 \end{bmatrix} = t \begin{bmatrix} 7 \\ -2 \\ 1 \\ 0 \end{bmatrix}$ (2), and a basis for the solution space is $\left\{ \begin{bmatrix} 7 \\ -2 \\ 1 \\ 0 \end{bmatrix} \right\}$. (2)

[4] d) Rank of A = dim (row space) = dim (col. space) = 3

[4] e) Nullity of A = dim (solution space) = 1.