Math 4443/5443
Exam 2

Instructions: Do problems 1, 2 and any five of the six remaining problems (for a total of 100 possible points).

1. *(20 points)* Give a proof of the following theorem: "Suppose f is a Riemann integrable function on $[a,b]$, and suppose F is a continuous function on $[a,b]$ such that $F'(x) = f(x)$ for all x in (a,b). Then $\int_a^b f = F(b) - F(a)$." (This is part 1 of the Fundamental Theorem of Calculus.)

2. *(20 points)* Give a proof of the following theorem: "Suppose (f_n) is a sequence of continuous functions on a set $A \subset \mathbf{R}$, and suppose (f_n) converges uniformly on A to a function $f : A \to \mathbf{R}$. Then f is continuous on A."

3. *(12 points)* Suppose $f : [0,1] \cup [2,3] \to \mathbf{R}$, and suppose f is uniformly continuous on $[0,1]$ and uniformly continuous on $[2,3]$. Prove that f is uniformly continuous on $[0,1] \cup [2,3]$.

4. *(12 points)* Define $f_n : [0,1] \to \mathbf{R}$ by

$$f_n(x) = \begin{cases} n^2x - n^3x^2 & \text{for } 0 \le x \le \frac{1}{n} \\ 0 & \text{for } \frac{1}{n} \le x \le 1. \end{cases}$$

a. Show that $\lim(f_n(x)) = 0$ for all $x \in [0,1]$.
b. Show that $\lim \int_0^1 f_n(x)\, dx \ne 0$.

5. *(12 points)* Prove that the sequence (f_n) defined in problem **4** does not converge uniformly on $[0,1]$ (you may assume the statements **a**, **b** in problem **4** are true).

6. *(12 points)* Either prove, or give a counterexample to, the following statement: "Suppose f is a continuous function on $\mathbf{R}$. Then there exists another function g on $\mathbf{R}$ such that $g'(x) = f(x)$ for all $x \in \mathbf{R}$.

7. *(12 points)* Let $F : [0,\infty) \to \mathbf{R}$ be defined by $F(x) = \displaystyle\int_0^x \sqrt{1 + (\sin^2 t)}\, dt$. Show that F is strictly increasing on $[0,\infty)$.

8. *(12 points)* Let $f_n(x) = 2^{-nx^2}$ for $x \in [-1,1]$.

a. Show that f_n converges pointwise on $[-1,1]$ to a function $f(x)$, and find $f(x)$.
b. Does f_n converge uniformly on $[-1,1]$? Prove your answer.