Math 4163
Exam 2

Name: Answer key

NOTE: On this exam, you may use any solution formula from class, without having to rederive it.

1. *(30 points)* Solve Laplace's equation inside a rectangle $0 \le x \le L$, $0 \le y \le H$, with the boundary conditions

$$u(0,y)=0, \qquad u(L,y)=0, \qquad u(x,0)=0, \qquad \frac{\partial u}{\partial y}(x,H)=f(x).$$

Write the answer as an infinite series, and express the coefficients in terms of the function $f(x)$.

Write $u(x,y) = P(x)\,Q(y)$ where $\dfrac{P''(x)}{P(x)} = -\lambda = \dfrac{-Q''(y)}{Q(y)}$, (4)

$P(0)=0$ (2), ~~and~~ $P(L)=0$ (2), and $Q(0)=0$ (2). The boundary-value problem

$$\begin{cases} P''(x) = -\lambda P(x) \\ P(0)=0 \\ P(L)=0 \end{cases}$$

has ~~all~~ eigenvalues $\lambda = +\left(\frac{n\pi}{L}\right)^2$ (2) and eigenfunctions

$P(x) = \sin\left(\frac{n\pi x}{L}\right)$ (2). The equation $Q''(y) = (+\lambda)\,Q(y) = \left(\frac{n\pi}{L}\right)^2 Q(y)$ (2)

has solutions $Q(y) = A\cosh\left(\frac{n\pi y}{L}\right) + B\sinh\left(\frac{n\pi y}{L}\right)$ (2), and from

$Q(0)=0$ we get $A=0$, so $Q(y) = B\sinh\left(\frac{n\pi y}{L}\right)$ (2). Therefore

separated solutions are $u_n(x,y) = B_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)$; $n=1,2,\dots$

Write $\boxed{u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\sinh\left(\frac{n\pi y}{L}\right)}$ (2). Then

$$\frac{\partial u}{\partial y}(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)\cdot\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi y}{L}\right) \text{ (2), and}$$

$$\frac{\partial u}{\partial y}(x,H) = \sum_{n=1}^{\infty} B_n\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi H}{L}\right)\sin\left(\frac{n\pi x}{L}\right) \text{ (2)} = f(x).$$

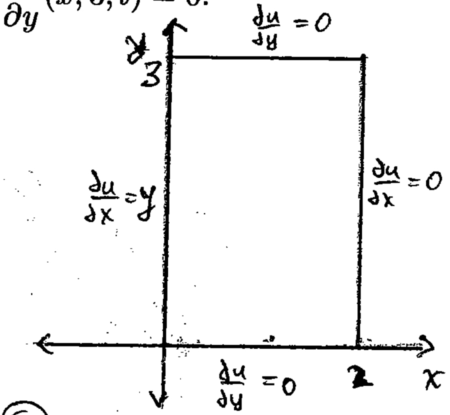
Therefore $B_n\cdot\left(\frac{n\pi}{L}\right)\cdot\cosh\left(\frac{n\pi H}{L}\right) = \frac{2}{L}\int_0^L f(w)\sin\left(\frac{n\pi w}{L}\right)dw$ (2), or

$$\boxed{B_n = \frac{2}{L\left(\frac{n\pi}{L}\right)\cosh\left(\frac{n\pi H}{L}\right)}\int_0^L f(w)\sin\left(\frac{n\pi w}{L}\right)dw} \text{ (2)}$$

2. *(15 points)*

[10] **a.** Suppose $u(x,y,t)$ is a solution of the equation $\frac{\partial u}{\partial t} = \nabla^2 u$ on the rectangle $0 \le x \le 2$, $0 \le y \le 3$. Suppose that at all times t, u satisfies the boundary conditions

$$\frac{\partial u}{\partial x}(0,y,t) = y, \qquad \frac{\partial u}{\partial x}(2,y,t) = 0, \qquad \frac{\partial u}{\partial y}(x,0,t) = 0, \qquad \frac{\partial u}{\partial y}(x,3,t) = 0.$$



(See diagram.) Find the derivative with respect to t of the integral

$$\boxed{\text{see next page for alternate solution}} \qquad \int_0^3 \int_0^2 u \, dx \, dy.$$

(Hint: use the divergence theorem.)

$$\frac{d}{dt} \int_0^3 \int_0^2 u \, dx\,dy = \int_0^3 \int_0^2 \frac{\partial u}{\partial t} \, dx\,dy \; \text{①} =$$

$$= \int_0^3 \int_0^2 \vec{\nabla}^2 u \, dx\,dy \; \text{①} = \int_0^3 \int_0^2 \vec{\nabla} \cdot (\vec{\nabla} u)\, dx\,dy = \oint_C (\vec{\nabla} u \cdot \vec{n}) \; \text{②},$$

where the integral $\oint_C$ is over the boundary of the rectangle and $\vec{n}$ is the outward normal to the rectangle. Then

$$\oint_C \vec{\nabla} u \cdot \vec{n} = \int_{\text{left edge}} \vec{\nabla} u \cdot (-\vec{i}) + \int_{\text{top edge}} \vec{\nabla} u \cdot \vec{j} + \int_{\text{right edge}} \vec{\nabla} u \cdot (+\vec{i}) + \int_{\text{bottom edge}} \vec{\nabla} u \cdot (-\vec{j}) \; \text{②} =$$

$$= \int_0^3 \left(-\frac{\partial u}{\partial x}(0,y)\right) dy + \int_0^2 \frac{\partial u}{\partial y}(x,3)\, dx + \int_0^3 \frac{\partial u}{\partial x}(2,y)\, dy + \int_0^2 -\frac{\partial u}{\partial y}(x,0)\, dx \; \text{②} =$$

$$= \int_0^3 (-y)\, dy + \int_0^2 0 \, dx + \int_0^3 0 \, dy + \int_0^2 0 \, dx = -\frac{9}{2} \; \text{②}, \text{ So } \boxed{\frac{d}{dt} \int_0^3 \int_0^2 u \, dx\,dy = -\frac{9}{2}.}$$

[5] **b.** Does the equation $\nabla^2 u = 0$ have a solution on the rectangle which satisfies the boundary conditions

$$\frac{\partial u}{\partial x}(0,y) = y, \qquad \frac{\partial u}{\partial x}(2,y) = 0, \qquad \frac{\partial u}{\partial y}(x,0) = 0, \qquad \frac{\partial u}{\partial y}(x,3) = 0?$$

Why or why not?

② No, because if $\nabla^2 u = 0$, then $u(x,y)$ is a time-independent solution of $\frac{\partial u}{\partial t} = \nabla^2 u$, with $\frac{\partial u}{\partial t} = 0$. But from part a) it would follow that $\frac{d}{dt} \int_0^3 \int_0^2 u \, dx\,dy = \int_0^3 \int_0^2 \frac{\partial u}{\partial t} \, dx\,dy = -\frac{9}{2}$ ①, if u satisfied the above boundary conditions. Since $\int_0^3 \int_0^2 \frac{\partial u}{\partial t} \, dx\,dy = \int_0^3 \int_0^2 0 \, dx\,dy = 0$, this is impossible. ②

Problem 2a, alternate solution:

Use the law of conservation of energy:

$$\frac{d}{dt}\int_0^3\int_0^2 e\, dx\, dy = -\oint_C \vec{\phi}\cdot\vec{n}.$$

Since $e = c\rho u$ and $\vec{\phi} = -K_0 \vec{\nabla} u$, this gives

$$\frac{d}{dt}\int_0^3\int_0^2 c\rho u\, dx\, dy = K_0 \oint_C \vec{\nabla} u\cdot\vec{n},$$

and so $$\frac{d}{dt}\int_0^3\int_0^2 u\, dx\, dy = \frac{K_0}{c\rho}\oint_C \vec{\nabla} u\cdot\vec{n} = k\oint_C \vec{\nabla} u\cdot\vec{n}$$

In this problem, $\frac{\partial u}{\partial t} = k\nabla^2 u = \nabla^2 u$, so $k=1$. Therefore

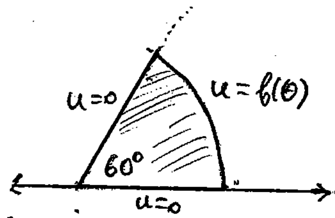
$$\frac{d}{dt}\int_0^3\int_0^2 u\, dx\, dy = \oint_C \vec{\nabla} u\cdot\vec{n}$$

From here we proceed as in the solution on the preceding page: we use the boundary conditions on u to find that $\oint_C \vec{\nabla} u\cdot\vec{n} = \frac{-9}{2}$,

so $$\frac{d}{dt}\int_0^3\int_0^2 u\, dx\, dy = -\frac{9}{2}.$$

3. *(25 points)* Consider the problem of solving Laplace's equation inside a 60^o wedge of radius a, subject to the boundary conditions

$$u(r,0) = 0, \qquad u(r,\frac{\pi}{3}) = 0, \qquad u(a,\theta) = f(\theta),$$



and the condition that u remain bounded near $r = 0$.

a. Separated solutions of Laplace's equations in polar coordinates have the form $u(r,\theta) = G(r)\phi(\theta)$, where $\frac{d^2\phi}{d\theta^2} = -\lambda\phi$. Given that u satisfies $u(r,0) = 0$ and $u(r,\frac{\pi}{3}) = 0$, find the eigenvalues λ and corresponding eigenfunctions $\phi(\theta)$. You may assume $\lambda > 0$.

If $\frac{d^2\phi}{d\theta^2} = -\lambda\phi$ and $\lambda>0$, then $\phi(\theta) = A\cos(\sqrt{\lambda}\,\theta) + B\sin(\sqrt{\lambda}\,\theta)$. (2)

Since $u(r,0) = u(r,\frac{\pi}{3}) = 0$, Then $\phi(0) = \phi(\frac{\pi}{3}) = 0$. But

$\phi(0) = 0 \Rightarrow 0 = A \Rightarrow \phi(\theta) = B\sin(\sqrt{\lambda}\,\theta)$ (2), and

$\phi(\frac{\pi}{3}) = 0 \Rightarrow 0 = B\sin(\sqrt{\lambda}\,\frac{\pi}{3})$. (1) For an eigenvalue we must have (1)

$\sin(\sqrt{\lambda}\frac{\pi}{3}) = 0$; so $\sqrt{\lambda}\,\frac{\pi}{3} = n\pi$ $(n = 1,2,3,\dots)$ (2) ; so $\sqrt{\lambda} = 3n$ or $\boxed{\lambda = (3n)^2}$.

Then $\boxed{\phi(\theta) = \sin(3n\theta)}$ (2)

b. The equation for $G(r)$ is $r^2\frac{d^2G}{dr^2} + r\frac{dG}{dr} - \lambda G = 0$, which has solutions $G = r^{\sqrt{\lambda}}$ and $G = r^{-\sqrt{\lambda}}$. Which of these solutions do we use in this problem, and why?

Here $\sqrt{\lambda} = 3n$ and the solutions are $G = r^{3n}$ and $G = r^{-3n}$.
But $u = G\cdot\phi$ must remain bounded near $r=0$. Since r^{3n} remains bounded near $r=0$ and r^{-3n} (3) does not, we use only the solution $G = r^{3n}$.
(2)

c. Write the solution u of the boundary-value problem as a sum of separated solutions, and express the coefficients in terms of the function $f(\theta)$.

Take $u(r,\theta) = \sum_{n=1}^{\infty} B_n r^{3n}\sin(3n\theta)$, (2) so

$$u(a,\theta) = \sum_{n=1}^{\infty} B_n a^{3n}\sin(3n\theta) = f(\theta), \quad \text{for } 0<\theta<\frac{\pi}{3}. \quad (2)$$

Taking $L = \frac{\pi}{3}$, we see that $\frac{n\pi}{L}\theta = 3n\theta$, so this series for $f(\theta)$ can be rewritten as

$$u(a,\theta) = \sum_{n=1}^{\infty} B_n a^{3n}\sin(\frac{n\pi}{L}\theta) = f(\theta), \quad \text{for } 0<\theta<L.$$

This is a Fourier sine series for $f(\theta)$ on $[0,L]$, so from the formula for coefficients of a Fourier sine series, we get $B_n a^{3n} = \frac{2}{L}\int_0^L f(w)\sin(\frac{n\pi}{L}w)\,dw$ (2) (2)

or $\boxed{B_n = \frac{2}{(\pi/3)a^{3n}}\int_0^{\pi/3} f(w)\sin(3nw)\,dw.}$ (2)

4. *(30 points)* Consider the function $f(x)$ defined on $[0,\pi]$ by

$$f(x) = \begin{cases} 1 \text{ for } 0 \le x \le \frac{\pi}{2} \\ 0 \text{ for } \frac{\pi}{2} \le x \le \pi. \end{cases}$$

a. Find the coefficients of the Fourier cosine series for f on $[0,\pi]$. Write out the first three non-zero terms of the series.

[20]

$$A_0 = \frac{1}{\pi}\int_0^\pi f(w)\,dw = \frac{1}{\pi}\int_0^{\frac{\pi}{2}} 1\,dw = \frac{1}{\pi}\cdot\frac{\pi}{2} = \boxed{\frac{1}{2}.}$$ (1) (2) (2) (2)

For $n \ge 1$, $A_n = \frac{2}{\pi}\int_0^\pi f(w)\cos(nw)\,dw = \frac{2}{\pi}\int_0^{\frac{\pi}{2}}\cos(nw)\,dw = \frac{2}{\pi}\left[\frac{\sin nw}{n}\right]_{w=0}^{w=\frac{\pi}{2}} =$ (1) (2) (2) (2)

$$= \boxed{\frac{2}{\pi n}\sin\left(\frac{n\pi}{2}\right)}$$ (2)

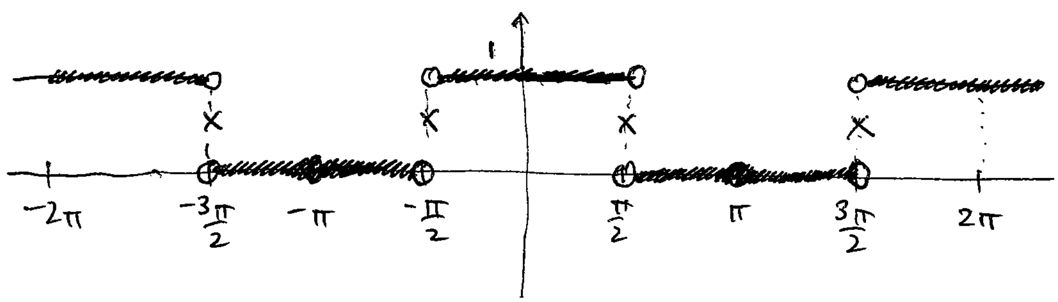
Thus $A_1 = \frac{2}{\pi}$, $A_2 = 0$, $A_3 = \frac{-2}{3\pi}$, $A_4 = 0$, etc.

We have (4)

$$f(x) \sim A_0 + \sum_{n\ge1}^{\infty} A_n\cos(nx) = \boxed{\frac{1}{2} + \frac{2}{\pi}\cos x - \frac{2}{3\pi}\cos 3x + \ldots}$$

[10]

b. Sketch the graph of the function to which the series in part **a** converges. Show the function on at least the interval $[-2\pi, 2\pi]$. Use an $\times$ to mark the values of the function at the points where it is discontinuous.



(The series converges to the even extension of f which is periodic of period 2π.) (3) (3)

(x's in correct spots : (4) pts.)