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Math 4433
Exam 2

Name: answer key

Unless otherwise stated, you may use any result from class without proof. But when using a major theorem, try to indicate which theorem you are using.

1. *(10 points)*

[5] **a.** Give the definition of an infinite series. Suppose (a_n) is a sequence. ① Then the infinite series Σa_n is defined to be the ~~sequence~~
① (s_n) given inductively by: $s_1 = a_1$ ①, and for all $k \geq 1$,
$s_{k+1} = s_k + a_k$. ②

[5] **b.** Give the definition of the limit of a function at a point.
Suppose $f : A \to \mathbb{R}$ and c is a cluster point of A. ①
We say that the limit of f at $x = c$ is equal to L if, for every
① $\varepsilon > 0$, there exists $\delta > 0$ ① such that if $0 < |x - c| < \delta$ ① then
$|f(x) - L| < \varepsilon$. ①

2. *(20 points)* State and prove the Monotone Convergence Theorem. (You need only consider increasing sequences.)

④ Theorem: If (x_n) is increasing and bounded above, then (x_n) converges.

Proof: ② Let $S = \{x_n : n \in \mathbb{N}\}$. Then S is nonempty (because $x_1 \in S$) and bounded (because (x_n) is bounded).
② So, by the Completeness Axiom, S has a supremum. Let
② $x = \sup S$. We will show that $\lim(x_n) = x$:

① Let $\varepsilon > 0$ be given. From the definition of supremum it follows that $x - \varepsilon$ is not an upper bound of S, so there
② exists $y \in S$ s.t. $x - \varepsilon < y$. Since $y \in S$, $y = x_K$ for some $K \in \mathbb{N}$. Suppose $n \geq K$. ① Since (x_n) is increasing, $x_n \geq x_K$. ②
So $x - \varepsilon < y = x_K \leq x_n$, so $x - \varepsilon < x_n$. ② Also, since $x_n \in S$ and $x = \sup S$, then $x_n \leq x$ ②. So $x_n < x + \varepsilon$. So $x - \varepsilon < x_n < x + \varepsilon$.

3. *(20 points)* Suppose the sequence (x_n) is defined inductively by letting $x_1 = 1$ and $x_{n+1} = \dfrac{2x_n + 11}{9}$ for all $n \in \mathbf{N}$. Prove that (x_n) converges and find its limit.

First we'll show (x_n) is increasing; i.e. $x_{k+1} \geq x_k$ for all $k \in \mathbb{N}$. We use induction. It is true for $k=1$, since $x_2 = \frac{13}{9}$ and $x_1 = 1$. (1) Now assume it is true for k. Then $x_{k+1} \geq x_k$

$$\Rightarrow \frac{2x_{k+1}+11}{9} \geq \frac{2x_k+11}{9} \overset{(4)}{\Rightarrow} x_{k+2} \geq x_{k+1};$$ so it is also true for $k+1$. (2)

Next we'll show (x_n) is bounded above by 2, (2) again using induction. For $k=1$ we have $x_1 = 1 \leq 2$. (1) Assume $x_k \leq 2$; then $x_{k+1} = \frac{2x_k+11}{9} \leq \frac{2\cdot 2+11}{9} = \frac{15}{9} \leq \frac{18}{9} = 2$, (4) so $x_{k+1} \leq 2$.

(2) Since (x_n) is increasing and bounded above, then by the MCT, (x_n) converges to some limit x. Then $\lim(x_{n+1}) = x$ also. So $x = \lim(x_{n+1}) = \lim\left(\frac{2x_n+11}{9}\right) = \frac{2x+11}{9}$ (2) (by Th 3.2.3).
So $x = \frac{2x+11}{9}$, or $9x = 2x+11$, or $7x = 11$, or $x = 11/7$. (2)

4. *(15 points)* Suppose the sequence (x_n) is defined inductively by letting $x_1 = 0$ and $x_{n+1} = x_n + (-1)^n n^3$ for all $n \in \mathbf{N}$.

[10] **a.** Prove that (x_n) is not a Cauchy sequence.

Let $\varepsilon = 1$. (2)

Let $K \in \mathbb{N}$ be given. (2)

Choose $n = K$ and $m = K+1$. (2) Then $m, n \geq K$, and $|x_m - x_n| \overset{(2)}{=} |x_{K+1} - x_K| = |(-1)^K K^3| = K^3 \geq 1 = \varepsilon$. (2)

This proves that there exists an $\varepsilon > 0$ such that for all $K \in \mathbb{N}$, there exist $m, n \geq K$ such that $|x_m - x_n| \geq \varepsilon$. So (x_n) is not Cauchy.

[5] **b.** Prove that (x_n) diverges.

In part a) we showed that (x_n) is not a Cauchy sequence. It therefore follows from the Cauchy criterion that (x_n) does not converge.

5. *(20 points)*

a. Prove that the series $\sum \frac{1}{1+2^n}$ converges.

[10] Let $(a_n) = \left(\frac{1}{1+2^n}\right)$. Then $0 < a_n < \frac{1}{2^n}$ ③ for all $n \in \mathbb{N}$, and, as shown in class, $\sum \frac{1}{2^n}$ converges. ③ It therefore follows from the Comparison Test ④ that $\sum (a_n)$ converges.

b. Prove that the series $\sum \frac{1}{1+(\frac{1}{2})^n}$ diverges.

[10] Let $(a_n) = \left(\frac{1}{1+(\frac{1}{2})^n}\right)$. By Theorem 3.2.3, ③ $\lim (a_n) = \frac{1}{\lim(1+(\frac{1}{2})^n)} = \frac{1}{1+\lim(\frac{1}{2})^n} = \frac{1}{1+0} = 1$. Since $\lim(a_n) = 1 \neq 0$, ③ then by the "n^{th}-term test", ④ $\sum a_n$ diverges.

6. *(15 points)* Use the ϵ-δ definition of limit to show that $\lim_{x\to 2} 3x = 6$.

Let $\varepsilon > 0$ be given.

⑤ Choose $\delta = \frac{\varepsilon}{3}$

⑤ If $0 < |x-2| < \delta$ then $|x-2| < \frac{\varepsilon}{3}$, so

⑤ $|3x-6| < \varepsilon$. This proves $\lim_{x\to 2} 3x = 6$!