Answers to Test 2

[20] 1. (2) Let $S = \{x_n : n \in \mathbb{N}\}$. Then S is non-empty and bounded (since (x_n) is bounded), so by the Completeness Axiom,
(2) S has a supremum $u \in \mathbb{R}$. Let $\varepsilon > 0$ be given. (1) Since
(1) $u - \varepsilon < u$, then from the definition of supremum it
(2) follows that there exists $y \in S$ such that $u - \varepsilon < y$.
(2) Since $y \in S$, there exists $K \in \mathbb{N}$ such that $y = x_K$.
(2) If $n \geq K$, then (2) $x_n \geq x_K$ (since (x_n) is increasing),
(2) so $u - \varepsilon < y = x_K \leq x_n$, so $u - \varepsilon < x_n$. Also, since
(2) $x_n \in S$ and $u = \sup S$, then $x_n \leq u$, so $x_n < u + \varepsilon$.
(2) Hence $u - \varepsilon < x_n < u + \varepsilon$, so $|x_n - u| < \varepsilon$. This proves $\lim(x_n) = u$, so (x_n) converges.

2. (a) We say (x_n) is Cauchy if for every $\varepsilon > 0$ (1)
[5] there exists (1) $H \in \mathbb{N}$ such that if (1) $m \geq H$ and $n \geq H$ (1), then $|x_m - x_n| < \varepsilon$ (1)

[15] (b) Suppose $\lim(x_n) = x$ ~~and (x_n) is Cauchy~~
(1) Let $\varepsilon > 0$ be given. Since $\lim(x_n) = x$ and $\frac{\varepsilon}{2} > 0$, (1)
(1) then there exists $K \in \mathbb{N}$ such that if $n \geq K$ (1) then $|x_n - x| < \varepsilon/2$. (1)
(2) If $m \geq K$ and $n \geq K$, then $|x_m - x| < \varepsilon/2$ and $|x_n - x| < \varepsilon/2$, (2)
so $|x_m - x_n| = |x_m - x + x - x_n| \leq |x_m - x| + |x - x_n| < \frac{\varepsilon}{2} + \varepsilon/2 = \varepsilon$.
(2) (2)

Therefore $|x_m - x_n| < \varepsilon$. This proves (x_n) is Cauchy.
(2)

[20] 3. First we use induction to show that $x_k \leq x_{k+1}$ for all $k \in \mathbb{N}$.
For $k = 1$ we have $x_k = x_1 = 1$ and $x_{k+1} = x_2 = \sqrt{6 + x_1} = \sqrt{7}$.
(2) Since $\sqrt{7} > 1$, then $x_2 \geq x_1$.
(2) Now assume $x_k \leq x_{k+1}$. Then $x_{k+1} = \sqrt{6 + x_k}$, and $6 + x_k \leq 6 + x_{k+1}$, so $\sqrt{6 + x_k} \leq \sqrt{6 + x_{k+1}}$, so $x_{k+1} \leq \sqrt{6 + x_{k+1}}$
(2) But $x_{k+2} = \sqrt{6 + x_{k+1}}$, so $x_{k+1} \leq x_{k+2}$. This completes the proof by induction.
Next we use induction to show that $x_k \leq 10$ for all $n \in \mathbb{N}$.
(2) For $k = 1$ we have $x_1 = 1 \leq 10$. (2) Assume $x_k \leq 10$; then
(2) $x_{k+1} = \sqrt{6 + x_k} \leq \sqrt{6 + 10} = \sqrt{16} = 4 \leq 10$, so $x_{k+1} \leq 10$.

We have proved that (x_k) is increasing and bounded
(2) above by 10, so by the MCT, (x_k) converges. Let x be

(cont'd)

the limit of (x_k). Since $x_{k+1} = \sqrt{6+x_k}$ for all $k \in \mathbb{N}$, we have from the Theorems of chapter 3 that
$\lim(x_{k+1}) = \lim \sqrt{6+x_k} = \sqrt{\lim(6+x_k)} = \sqrt{\lim(6)+\lim(x_k)} =$
$= \sqrt{6+x}$, so $\lim(x_{k+1}) = \sqrt{6+x}$. But $\lim(x_{k+1}) =$
(2) $= \lim(x_k) = x$, so $x = \sqrt{6+x}$. Hence $x^2 = 6+x$,
(2) so $x^2 - x - 6 = 0$, so $(x-3)(x+2) = 0$, so $x = 3$ or $x = -2$.
(2) But since $x_1 = 1$ and (x_k) is increasing, then $x_k \geq 1$ for all $k \in \mathbb{N}$, so $x = \lim(x_k) \geq 1$. Hence $x = 3$.

4. (a) [10] For every $k \in \mathbb{N}$, $y_k = \max\{x_1, x_2, \ldots, x_k\}$, so y_k is equal to one of the numbers in $\{x_1, x_2, \ldots, x_k\}$. Hence y_k is also in the set $\{x_1, x_2, \ldots, x_k, x_{k+1}\}$. Since y_{k+1} is the largest element of the set $\{x_1, x_2, \ldots, x_{k+1}\}$, it follows that y_{k+1} is larger than (or equal to) y_k. This proves that (y_k) is an increasing sequence.

Also, we are given there exists $M \in \mathbb{R}$ such that $\forall n \in \mathbb{N}$, $|x_n| \leq M$. For every $k \in \mathbb{N}$, y_k is one of the numbers in $\{x_1, \ldots, x_k\}$, so it follows that $|y_k| \leq M$. This proves that (y_k) is bounded.

Since (y_k) is bounded and increasing, (y_k) converges by the MCT

4b. [see below]

[10] 5. Let $\varepsilon > 0$ be given. (1) Let $\delta = \frac{\varepsilon}{5}$. (2) If $0 < |x-7| < \delta$, (1) then $|x-7| < \varepsilon/5$, so $5|x-7| < \varepsilon$, (2) so $|5x-35| < \varepsilon$, (2) so $|(5x-3) - 32| < \varepsilon$. (2)

[10] 6. (first proof) Let $\varepsilon > 0$ be given. (1) Choose $\delta > 0$ such that if $0 < |x-c| < \delta$ (2) then $|g(x) - 0| < \varepsilon$. If $0 < |x-c| < \delta$, (1) then
(2) $|f(x)g(x) - 0| = |f(x)||g(x)| \leq 1 \cdot |g(x)| = |g(x) - 0| < \varepsilon$. (2) (2)
So $\lim_{x \to c} (f(x)g(x)) = 0$

(second proof) Let (x_n) be given such that $x_n \neq c$ for all $n \in \mathbb{N}$ and $\lim(x_n) = c$. (2) Then $|f(x_n)| \leq 1$ (2) for all $n \in \mathbb{N}$, and $\lim(g(x_n)) = 0$, (2) by the sequential criterion. From a homework problem it follows that $\lim(f(x_n)g(x_n)) = 0$. (2) So by the sequential criterion, $\lim_{x \to c} f(x)g(x) = 0$. (2)

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[10] 4b. Let $(x_n) = (\frac{1}{n})$. Then for all $k \in \mathbb{N}$, $y_k = \max\{1, \frac{1}{2}, \ldots, \frac{1}{k}\} = 1$.
So $(y_k) = (1, 1, 1, 1, \ldots)$, which is not a subsequence of $(\frac{1}{n})$!
So the statement is false.