Answers to problems on test 2

3. Let $\varepsilon > 0$ be given. By assumption, there exists $\delta_1 > 0$ (2) such that if $x, y \in [0,1]$ and $|x-y| < \delta_1$, then $|f(x) - f(y)| < \varepsilon$. Also, there exists $\delta_2 > 0$ s.t. if $x, y \in [2,3]$ and $|x-y| < \delta_2$, then $|f(x)-f(y)| < \varepsilon$. (2)

Define $\delta = \min(\delta_1, \delta_2, 1)$ (2). Suppose $x, y \in [0,1] \cup [2,3]$ and $|x-y| < \delta$. Then $|x-y| < 1$; so we can't have $x \in [0,1]$ and $y \in [2,3]$ (because that would mean $y - x \geq 2 - 1 = 1$) and we can't have $x \in [2,3]$ and $y \in [0,1]$ (because that would mean $x - y \geq 2 - 1 = 1$). (2) So either (i) x & y are both in $[0,1]$ or (ii) x, y are both in $[2,3]$. In case (i), then since $|x-y| < \delta \leq \delta_1$, we have $|f(x)-f(y)| < \varepsilon$ (2); and in case (ii), since $|x-y| < \delta \leq \delta_2$, we have $|f(x)-f(y)| < \varepsilon$. (2)

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4. a. If $x = 0$, then $f_n(x) = f_n(0) = 0$ for all n, (1) so $\lim(f_n(x)) = 0$.
If $x > 0$, then choose $N \in \mathbb{N}$ s.t. $N > \frac{1}{x}$. (2) For all $n \geq N$, we have $\frac{1}{n} \leq \frac{1}{N} < x$, so $f_n(x) = 0$. (2) It follows that $\lim(f_n(x)) = 0$. (1) (1)

b.
$$\lim\left(\int_0^1 f_n(x)\,dx\right) = \lim\left(\int_0^{\frac{1}{n}} (n^2 x - n^3 x^2)\,dx + \int_{\frac{1}{n}}^1 0\,dx\right) =$$
$$= \lim\left(n^2 \int_0^{\frac{1}{n}} x\,dx - n^3 \int_0^{\frac{1}{n}} x^2\,dx\right) = \lim\left(n^2 \left[\frac{x^2}{2}\right]_0^{\frac{1}{n}} - n^3 \left[\frac{x^3}{3}\right]_0^{\frac{1}{n}}\right)$$
$$= \lim\left(n^2 \cdot \frac{1}{2n^2} - n^3 \cdot \frac{1}{3n^3}\right) = \lim\left(\frac{1}{2} - \frac{1}{3}\right) = \left(\frac{1}{2} - \frac{1}{3}\right) = \frac{1}{6} \neq 0.$$

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5. Since f_n is continuous on $[0,1]$ for each n, (4) and f_n converges pointwise to 0 on $[0,1]$ by 4(a), then if f_n were to converge uniformly on $[0,1]$ it would follow from (4) a theorem proved in class that $\lim \int_0^1 f_n = \int_0^1 0 = 0$. (4) Since $\lim \int_0^1 f_n \neq 0$ by 4(b), then f_n cannot converge uniformly on $[0,1]$.

6. We'll prove the statement. Define $g: \mathbb{R} \to \mathbb{R}$ by
4 $g(x) = \left(\int_0^x f\right)$ for $x \in \mathbb{R}$. Let $x \in \mathbb{R}$ be given; we'll show that $g'(x) = f(x)$. Choose an interval $[a,b]$ such that $a < x < b$, Then $g(x) = \int_0^x f = \int_a^x f + \int_0^a f$.

Since f is continuous on $[a,b]$, then by the FTC part 2,
4 $\frac{d}{dx}\left(\int_a^x f\right) = f(x)$. Hence $g'(x) = \frac{d}{dx}\int_a^x f + \frac{d}{dx}\int_0^a f = f(x) + 0 = f(x)$.
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7. By the FTC part 2 (4), $F'(x) = \sqrt{1+\sin^2 x}$ (2) $\geq \sqrt{1} > 0$ (2) for all $x > 0$. It follows from a theorem proved in class (actually, in a homework problem) (4) that F is strictly increasing on $[0,\infty)$.

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8. a) If $x = 0$, then $f_n(x) = 2^{-n\cdot 0} = 2^0 = 1$ (2) for all n, so $\lim f_n(x) = 1$.
If $x \neq 0$, then $2^{-nx^2} = \frac{1}{(2^{x^2})^n} = b^n$ (3) where

$b = \frac{1}{2^{x^2}} < \frac{1}{2^0} = 1$, so $\lim b^n = 0$. Thus $\lim f_n(x) = 0$.

This shows that $\lim f_n(x) = f(x) = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{if } x \neq 0. \end{cases}$ (1)

b.) No (2), f_n cannot converge uniformly on $[-1,1]$; because if f_n were to converge uniformly to f on $[-1,1]$, then since each f_n is continuous on $[-1,1]$,
2 it would follow that f is continuous on $[-1,1]$.
But the function f we found in part a) is not
2 continuous on $[-1,1]$ (it is not cont. at 0.)