EXAM 3
Math 3333
4-19-07

Name key

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(10 points)* Define $L : R^2 \to R^2$ by $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a^2 \\ 3ab \end{bmatrix}$. Show that L is not a linear transformation. 3

We have $L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ 3 and $L\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = L\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, which is not equal to $2L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Since $L\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \neq 2L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$ Then L is not a linear transformation. 4

2. *(20 points)* Find an orthonormal basis for the subspace W of R^3 spanned by

$$\left\{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}\right\}.$$

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$, and apply Gram-Schmidt to $\{\vec{u}_1, \vec{u}_2\}$ to get an orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$ for W. We get $\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$ 2 and $\vec{v}_2 = \vec{u}_2 + a\vec{v}_1$ 2, where $(\vec{v}_2, \vec{v}_1) = 0$ 2 implies $(\vec{u}_2, \vec{v}_1) + a(\vec{v}_1, \vec{v}_1) = 0$ 2, or

$\begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} + a\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = 0$, or $18 + 9a = 0$, or $a = -2$. 2 So

$\vec{v}_2 = \vec{u}_2 - 2\vec{v}_1 = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}$. 2 Thus $\{\vec{v}_1, \vec{v}_2\} = \left\{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}\right\}$ is an orthogonal basis.

To get an orthonormal basis we divide each vector in $\{\vec{v}_1, \vec{v}_2\}$ by its length. 2 This gives $\vec{w}_1 = \frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}}{\sqrt{1^2+2^2+2^2}}$ 2 $= \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$ and

$\vec{w}_2 = \frac{\vec{v}_2}{\|\vec{v}_2\|}$ 2 2 $= \frac{\begin{bmatrix} 2 \\ 2 \\ -3 \end{bmatrix}}{\sqrt{2^2+2^2+(-3)^2}} = \begin{bmatrix} 2/\sqrt{17} \\ 2/\sqrt{17} \\ -3/\sqrt{17} \end{bmatrix}$. An orthonormal basis is $\left\{\begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}, \begin{bmatrix} 2/\sqrt{17} \\ 2/\sqrt{17} \\ -3/\sqrt{17} \end{bmatrix}\right\}$.

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is

3. (~~10~~ *points*) Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ ~~are~~ is an orthonormal set of vectors in an inner product space V and suppose that $\mathbf{v} = 2\mathbf{v}_1 - 3\mathbf{v}_3$. Find $(\mathbf{v}, \mathbf{v}_1)$, $(\mathbf{v}, \mathbf{v}_2)$, and $(\mathbf{v}, \mathbf{v}_3)$. Give a reason for your answer.

2 2

$(\vec{v}, \vec{v}_1) = (2\vec{v}_1 - 3\vec{v}_3, \vec{v}_1) = 2(\vec{v}_1, \vec{v}_1) - 3(\vec{v}_3, \vec{v}_1)$. Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthonormal then $(\vec{v}_1, \vec{v}_1) = 1$ and $(\vec{v}_3, \vec{v}_1) = 0$, so $(\vec{v}, \vec{v}_1) = 2 \cdot 1 - 3 \cdot 0 = \boxed{2}$

2 2 1

Similarly,

3 $(\vec{v}, \vec{v}_2) = (2\vec{v}_1 - 3\vec{v}_3, \vec{v}_2) = 2(\vec{v}_1, \vec{v}_2) - 3(\vec{v}_3, \vec{v}_2) = 2 \cdot 0 - 3 \cdot 0 = \boxed{0}$

and

3 $(\vec{v}, \vec{v}_3) = (2\vec{v}_1 - 3\vec{v}_3, \vec{v}_3) = 2(\vec{v}_1, \vec{v}_3) - 3(\vec{v}_3, \vec{v}_3) = 2 \cdot 0 - 3 \cdot 1 = \boxed{-3.}$

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4. (~~10~~ *points*) Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis vectors for R^3 and suppose

$$L(\mathbf{e}_1) = \begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix}, \quad L(\mathbf{e}_2) = \begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix}, \quad L(\mathbf{e}_3) = \begin{bmatrix} 1 \\ 11 \\ 4 \end{bmatrix}.$$

Find $L\left(\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right)$.

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$$L\left(\begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}\right) = L\left((-1)\vec{e}_1 + 2\vec{e}_2 + 1\vec{e}_3\right) = (-1)L(\vec{e}_1) + 2L(\vec{e}_2) + L(\vec{e}_3)$$

$$= (-1)\begin{bmatrix} 2 \\ 0 \\ 5 \end{bmatrix} + 2\begin{bmatrix} 3 \\ 6 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 11 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} -2 + 6 + 1 \\ 0 + 12 + 11 \\ -5 + 2 + 4 \end{bmatrix} = \begin{bmatrix} 5 \\ 23 \\ 1 \end{bmatrix}.$$

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5. (~~10~~ *points*) Suppose that $\mathbf{u}$ and $\mathbf{v}$ are vectors in R^n and A is an $n \times n$ matrix. Suppose that $A\mathbf{u}$ and $\mathbf{v}$ are orthogonal with respect to the standard inner product on R^n. Show that $\mathbf{u}$ and $A^T\mathbf{v}$ are orthogonal.

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6. *(30 points)* Define $L : R^3 \to R^4$ by $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 2u_2 - u_3 \\ 2u_1 + u_2 + 4u_3 \\ 3u_2 - 6u_3 \\ u_1 + u_2 + u_3 \end{bmatrix}$. Find

a) A basis for ker L.

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The standard matrix representation of L is $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & 4 \\ 0 & 3 & -6 \\ 1 & 1 & 1 \end{bmatrix}$,

2 2

and ~~the~~ ker L equals the null space of A. Row-reducing $\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 2 & 1 & 4 & 0 \\ 0 & 3 & -6 & 0 \\ 1 & 1 & 1 & 0 \end{array}\right]$ gives

3 3 2 1

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & -3 & 6 & 0 \\ 0 & 3 & 6 & 0 \\ 0 & -1 & 2 & 0 \end{array}\right] \to \left[\begin{array}{ccc|c} 1 & 2 & -1 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \to \begin{array}{l} u_1 + 2u_2 - u_3 = 0 \\ u_2 - 2u_3 = 0 \end{array}$$

So $u_3 = t$, 1

$u_2 = 2t$, 1

$u_1 = -2u_2 + u_3$

$= -4t + t = -3t$, 1

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and $\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} -3t \\ 2t \\ t \end{bmatrix} = t\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}$. So a basis for ~~range~~ ker L is $\left\{\begin{bmatrix} -3 \\ 2 \\ 1 \end{bmatrix}\right\}$. 1

1 1

b) A basis for range L.

range L is the column space of A. To find ~~the~~ a basis we row-reduce A as above to get $\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Only the

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1st and 2nd columns in the reduced matrix have initial 1's, so the 1st and 2nd columns of A form a basis for the column space:

$\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right\}$. 3