EXAM 3 Name ~~key~~

Math 3333

4-19-07

Name: key

__Instructions__ *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(10 points)* Define $L : R^2 \to R^2$ by $L\left(\begin{bmatrix} a \\ b \end{bmatrix}\right) = \begin{bmatrix} a^2 \\ 3ab \end{bmatrix}$. Show that L is not a linear transformation. 3 3 [redacted]

We have $L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ [redacted] and $L\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = L\left(\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 6 \end{bmatrix}$, which is not equal to $2L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. Since $L\left(2\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) \neq 2L\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$, then L is not a linear transformation. [redacted] 4

2. *(20 points)* Find an orthonormal basis for the subspace W of R^3 spanned by

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 10 \end{bmatrix} \right\}.$$

Let $\vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ and $\vec{u}_2 = \begin{bmatrix} 4 \\ 1 \\ 10 \end{bmatrix}$. We use Gram-Schmidt to find an orthogonal basis $\{\vec{v}_1, \vec{v}_2\}$. Let $\vec{v}_1 = \vec{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$ (2), and take

$\vec{v}_2 = \vec{u}_2 + a\vec{v}_1$ (2). From $(\vec{v}_2, \vec{v}_1) = 0$ (2) we get [redacted] $(\vec{u}_2 + a\vec{v}_1, \vec{v}_1) = 0$ or

$(\vec{u}_2, \vec{v}_1) + a(\vec{v}_1, \vec{v}_1) = 0$ (2), or $33 + a \cdot 11 = 0$, or $a = -3$. (2)

So $\vec{v}_2 = \begin{bmatrix} 4 \\ 1 \\ 10 \end{bmatrix} - 3\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}$. (2) Thus $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix} \right\}$ is an orthogonal basis.

To get an orthonormal basis we divide each vector in the orthogonal basis by its length. This gives $\vec{w}_1 = \dfrac{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}{\sqrt{11}}$ (2) (2) $= \begin{bmatrix} 1/\sqrt{11} \\ -1/\sqrt{11} \\ 3/\sqrt{11} \end{bmatrix}$

and $\vec{w}_2 = \dfrac{\begin{bmatrix} 1 \\ 4 \\ 1 \end{bmatrix}}{\sqrt{18}}$ (2) (2) $= \begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix}$. So an orthonormal basis is $\left\{ \begin{bmatrix} 1/\sqrt{11} \\ -1/\sqrt{11} \\ 3/\sqrt{11} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{18} \\ 4/\sqrt{18} \\ 1/\sqrt{18} \end{bmatrix} \right\}$.

3. (~~14~~ 15 *points)* Suppose $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are an orthonormal set of vectors in an inner product space V and suppose that $\mathbf{v} = 5\mathbf{v}_1 + 2\mathbf{v}_3$. Find $(\mathbf{v}, \mathbf{v}_1)$, $(\mathbf{v}, \mathbf{v}_2)$, and $(\mathbf{v}, \mathbf{v}_3)$. Give a reason for your answer.

$(\vec{v}, \vec{v}_1) = (5\vec{v}_1 + 2\vec{v}_3, \vec{v}_1) = 5(\vec{v}_1, \vec{v}_1) + 2(\vec{v}_3, \vec{v}_1)$. Since $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is orthonormal, then $(\vec{v}_1, \vec{v}_1) = 1$ and $(\vec{v}_3, \vec{v}_1) = 0$; so $(\vec{v}, \vec{v}_1) = 5\cdot 1 + 2\cdot 0 = 5$.

Similarly,

$(\vec{v}, \vec{v}_2) = (5\vec{v}_1 + 2\vec{v}_3, \vec{v}_2) = 5(\vec{v}_1, \vec{v}_2) + 2(\vec{v}_3, \vec{v}_2) = 5\cdot 0 + 2\cdot 0 = 0$

and $(\vec{v}, \vec{v}_3) = (5\vec{v}_1 + 2\vec{v}_3, \vec{v}_3) = 5(\vec{v}_1, \vec{v}_3) + 2(\vec{v}_3, \vec{v}_3) = 5\cdot 0 + 2\cdot 1 = 2$.

4. (~~10~~ 15 *points)* Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the standard basis vectors for R^3 and suppose

$$L(\mathbf{e}_1) = \begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix}, \quad L(\mathbf{e}_2) = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \quad L(\mathbf{e}_3) = \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix}.$$

Find $L\left(\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}\right)$.

$$L\left(\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}\right) = L\left((-1)\vec{e}_1 + 3\vec{e}_2 + \vec{e}_3\right) = (-1)L(\vec{e}_1) + 3L(\vec{e}_2) + L(\vec{e}_3) =$$

$$= (-1)\begin{bmatrix} -1 \\ 2 \\ 6 \end{bmatrix} + 3\begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 7 \\ 1 \\ 5 \end{bmatrix} = \begin{bmatrix} 17 \\ -1 \\ 11 \end{bmatrix}.$$

5. (~~14~~ 10 *points)* Suppose $L : V \to W$ is a linear transformation and $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are vectors in V. Suppose $\{L(\mathbf{v}_1), L(\mathbf{v}_2), L(\mathbf{v}_3)\}$ is linearly independent. Show that $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is linearly independent.

Suppose $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$.

Then $L(a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3) = L(\vec{0})$. But $L(\vec{0}) = \vec{0}$ as shown in class.

So $L(a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3) = \vec{0}$.

So $a_1 L(\vec{v}_1) + a_2 L(\vec{v}_2) + a_3 L(\vec{v}_3) = \vec{0}$ (by properties of linear transformations).

But since $\{L(\vec{v}_1), L(\vec{v}_2), L(\vec{v}_3)\}$ is independent it follows that $a_1 = a_2 = a_3 = 0$. This proves that $a_1\vec{v}_1 + a_2\vec{v}_2 + a_3\vec{v}_3 = \vec{0}$ can only happen if $a_1 = a_2 = a_3 = 0$, so $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

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6. *(30 points)* Define $L : R^4 \to R^3$ by $L\left(\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}\right) = \begin{bmatrix} u_1 + 3u_2 + 3u_3 - u_4 \\ 2u_1 + 10u_4 \\ 2u_1 + 4u_2 + 4u_3 + 2u_4 \end{bmatrix}$. Find

a) A basis for ker L.

The standard matrix representing L is $A = \begin{bmatrix} 1 & 3 & 3 & -1 \\ 2 & 0 & 0 & 10 \\ 2 & 4 & 4 & 2 \end{bmatrix}$. 4

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Then ker L is equal to the nullspace of A. To find this, we solve $A\vec{u} = \vec{0}$

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by row-reduction: $\left[\begin{array}{cccc|c} 1 & 3 & 3 & -1 & 0 \\ 2 & 0 & 0 & 10 & 0 \\ 2 & 4 & 4 & 2 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 3 & -1 & 0 \\ 0 & -6 & -6 & 12 & 0 \\ 0 & -2 & -2 & 4 & 0 \end{array}\right] \to \left[\begin{array}{cccc|c} 1 & 3 & 3 & -1 & 0 \\ 0 & 1 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$

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So $u_1 + 3u_2 + 3u_3 - u_4 = 0$ 2 We have $u_4 = t$, $u_3 = s$ (free parameters),
$u_2 + u_3 - 2u_4 = 0$.

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$u_2 = 2u_4 - u_3 = 2t - s$, and $u_1 = u_4 - 3u_3 - 3u_2 = t - 3s - 3(2t - s) = -5t$. So

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the nullspace of A is $\left\{ \begin{bmatrix} -5t \\ 2t-s \\ s \\ t \end{bmatrix} : s, t \in \mathbb{R} \right\} = \left\{ s\begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \end{bmatrix} : s, t \in \mathbb{R} \right\}$, and a

basis is $\left\{ \begin{bmatrix} 0 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -5 \\ 2 \\ 0 \\ 1 \end{bmatrix} \right\}$. 2

b) A basis for range L.

The range of L is the same as the column space of A, which is found by identifying the columns in the reduced form of A that ~~consist those~~ contain initial 1's and taking the corresponding columns of A. Since in $\begin{bmatrix} 1 & 3 & 3 & -1 \\ 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ only the first and second

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columns have initial 1's, then $\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix} \right\}$ is a basis for the

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range of L.