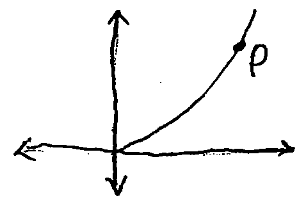
EXAM 3
Univ 2000
5-1-09

Name: ____________________

Instructions *Work all of the following problems in the space provided. If there is not enough room, you may write on the back sides of the pages. Give thorough explanations to receive full credit.*

1. *(20 points)* A curve is given parametrically by $x = t + t^3$, $y = t + t^4$.



a) Find the slope $\frac{dy}{dx}$ of the curve at the point P where $t = 1$ (see figure).

[8] $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1+4t^3}{1+3t^2}$ 2 2 2

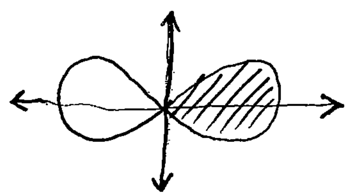
When $t=1$, $\frac{dy}{dx} = \frac{1+4}{1+3} = \frac{5}{4}$ 2

b) Find $\frac{d^2y}{dx^2}$ at the point P where $t = 1$.

[12] $\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{d}{dt}\left(\frac{1+4t^3}{1+3t^2}\right)}{(1+3t^2)} = \frac{\frac{(1+3t^2)\cdot 12t^2 - (1+4t^3)6t}{(1+3t^2)^2}}{(1+3t^2)}$ 2 2 2 4

When $t=1$, $\frac{d^2y}{dx^2} = \frac{\left[\frac{(1+3)\cdot 12 - (1+4)\cdot 6}{4^2}\right]}{4} = \frac{\left(\frac{48-30}{16}\right)}{4} = \frac{18}{64} = \frac{9}{32}$ 2

2. *(12 points)* Find the area of one loop of the curve given by the polar equation $r^2 = \cos 2\theta$ (shaded region at right).



$r = 0$ when $\cos 2\theta = 0$, or $2\theta = \pm\frac{\pi}{2}$, 2

or $\theta = \pm\frac{\pi}{4}$. 2 So the loop consists of points on the curve corresponding to $-\frac{\pi}{4} \le \theta \le \frac{\pi}{4}$. 2

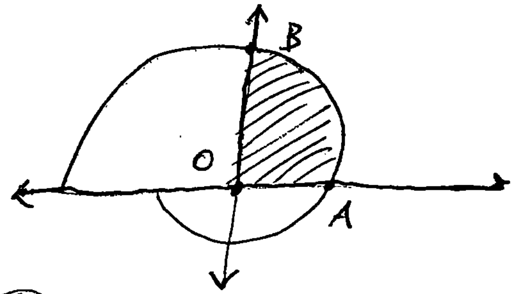
Then $A = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} r^2\, d\theta = \frac{1}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \cos 2\theta\, d\theta = \left[\frac{\sin 2\theta}{4}\right]_{-\pi/4}^{\pi/4}$ 2 2

$= \frac{1}{4}\left(\sin\frac{\pi}{2} - \sin\left(-\frac{\pi}{2}\right)\right) = \frac{1}{4}\left(1-(-1)\right) = \frac{2}{4} = \frac{1}{2}$. 2

 Name: ______________________

3. *(20 points)* A curve is given by the polar equation $r = e^{\theta}$ for $-\pi \le \theta \le \pi$ (see figure).

a) Find the area of the shaded region OAB.



[10]

$$A = \frac{1}{2}\int_0^{\frac{\pi}{2}} r^2\, d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} (e^{\theta})^2\, d\theta = \frac{1}{2}\int_0^{\frac{\pi}{2}} e^{2\theta}\, d\theta$$

$$= \left[\frac{e^{2\theta}}{4}\right]_0^{\frac{\pi}{2}} = \frac{e^{\pi}}{4} - \frac{1}{4} = \boxed{\frac{1}{4}(e^{\pi}-1)}.$$

b) Find the arclength of the curve from A to B.

[10]

$$L = \int_0^{\frac{\pi}{2}} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta =$$

$$= \int_0^{\frac{\pi}{2}} \sqrt{2e^{2\theta}}\, d\theta = \int_0^{\frac{\pi}{2}} \sqrt{2}\, e^{\theta}\, d\theta = \left[\sqrt{2}\, e^{\theta}\right]_0^{\pi/2}$$

$$= \boxed{\sqrt{2}\,(e^{\frac{\pi}{2}} - 1)}.$$

4. *(12 points)* Find the sum of the series, if it exists.

a) $5 + \frac{5}{7^2} + \frac{5}{7^4} + \frac{5}{7^6} + \ldots$

[6]

$$S = 5\left(1 + \frac{1}{7^2} + \left(\frac{1}{7^2}\right)^2 + \left(\frac{1}{7^2}\right)^3 + \ldots\right) = 5\,\frac{1}{1-\left(\frac{1}{7^2}\right)} = \frac{5}{1-\frac{1}{49}}$$

$$\left[1 + r + r^2 + r^3 + \ldots = \frac{1}{1-r}\right] \qquad = \frac{5}{48/49} = \boxed{\frac{5\cdot 49}{48}} = \frac{245}{48}.$$

b) $\frac{11^2}{12^2} - \frac{11^3}{12^3} + \frac{11^4}{12^4} - \frac{11^5}{12^5} + \ldots$

[6]

$$S = \frac{11^2}{12^2}\left(1 - \frac{11}{12} + \left(\frac{11}{12}\right)^2 - \left(\frac{11}{12}\right)^3 + \ldots\right)$$

$$= \frac{11^2}{12^2}\left(1 + \left(-\frac{11}{12}\right) + \left(-\frac{11}{12}\right)^2 + \left(-\frac{11}{12}\right)^3 + \ldots\right) = \frac{11^2}{12^2}\,\frac{1}{1-\left(-\frac{11}{12}\right)} = \frac{11^2}{12^2}\left(\frac{1}{1+\frac{11}{12}}\right)$$

$$= \frac{11^2}{12^2}\,\frac{1}{23/12} = \boxed{\frac{11^2}{12^2}\cdot\frac{12}{23}} = \frac{121}{276}$$

Name: ______________________

5. *(12 points)* Use the integral test to determine whether the series $\sum_{n=2}^{\infty} \frac{1}{n \ln n [\ln(\ln n)]}$ converges. (Hint: try the substitution $u = \ln(\ln x)$.)

$$\int_2^{\infty} \frac{1}{x \ln x [\ln(\ln x)]} dx \; (2) = \lim_{b\to\infty} \int_2^b \frac{1}{x \ln x [\ln(\ln x)]} dx \; (2)$$

(2) $\boxed{\begin{array}{l} u = \ln(\ln(x)) \\ du = \frac{1}{\ln x} \cdot \frac{1}{x} dx \end{array}}$

$$= \lim_{b\to\infty} \int_{\ln[\ln 2]}^{\ln[\ln(b)]} \frac{1}{u} du \; (2)$$

$$= \lim_{b\to\infty} \Big[\ln u\Big]_{u=\ln(\ln 2)}^{u=\ln[\ln(b)]} \; (2) = \lim_{b\to\infty} \{\ln[\ln(\ln b)] - \ln[\ln(\ln 2)]\}$$

$= \infty$ (2), so the series diverges.

6. *(24 points)* Decide whether the following series converge. Give reasons for your answers.

a) $\sum_{n=1}^{\infty} \frac{3^n}{3^n + 2^n}$

[8]

$$\lim_{n\to\infty} \frac{3^n}{3^n+2^n} = \lim_{n\to\infty} \frac{1}{1+\frac{2^n}{3^n}} \; (2) = \lim_{n\to\infty} \frac{1}{1+(\frac{2}{3})^n} = \frac{1}{1+0} = 1 \; (2)$$

(2)

Since the limit is not zero, the series diverges by the n^{th} term test. (2)

b) $\sum_{n=1}^{\infty} \frac{\sqrt{1+\sin^2 n}}{n}$

[8]

$\frac{\sqrt{1+\sin^2 n}}{n} \geq \frac{1}{n}$ (4), and $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, (2)

so the series diverges by the comparison test. (2)

c) $\sum_{n=1}^{\infty} \frac{1}{n^3 - 5n - 100}$

[8]

$\sum_{n=1}^{\infty} \frac{1}{n^3}$ (2) converges, and

$$\lim_{n\to\infty} \frac{\left(\frac{1}{n^3}\right)}{\left(\frac{1}{n^3-5n-100}\right)} \; (2) = \lim_{n\to\infty} \frac{n^3-5n-100}{n^3} = \lim_{n\to\infty} \frac{1-\frac{5}{n^2}-\frac{100}{n^3}}{1}$$

(2)

$= 1-0-0 = 1$, which is neither 0 nor ∞, so the series converges by the Limit Comparison Test. (2)