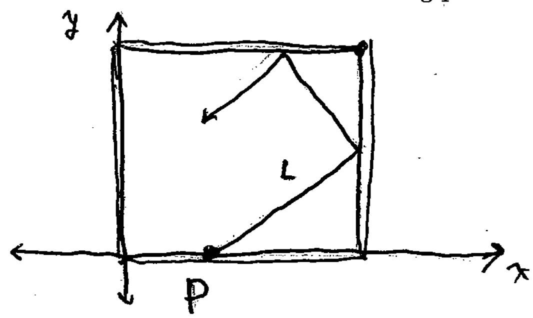
Math 4513
Take-home final exam

Do up to eight of these problems. Try to get six correct for an A, or three correct for a B. You should not collaborate with other students.

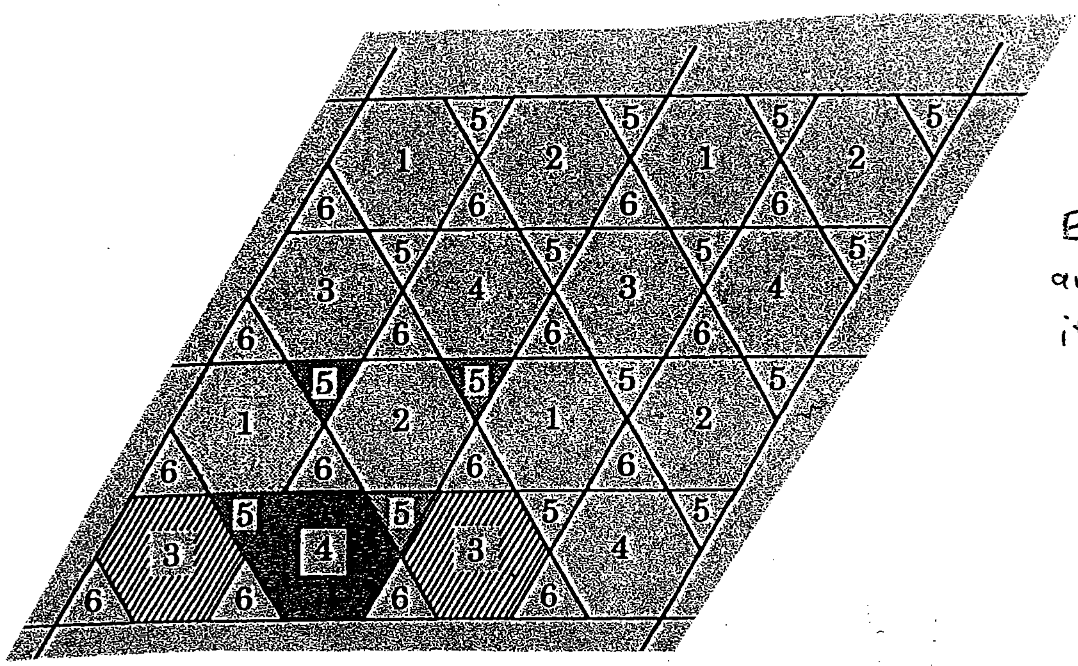
1. (Bauer/Rollins) A billiard table is in the form of a square with vertices at $(0,0)$, $(1,0)$, $(0,1)$, and $(1,1)$. A billiard ball is launched from a point P on the x-axis along a line L with slope m. Whenever the ball hits an edge of the table it bounces off with an angle of reflection equal to its angle of incidence. Prove that if m is irrational then the ball never returns to its starting point P.



2. (Beebe/Stevens) In the figure, the regions labelled 1 are red, the regions labelled 2 are yellow, the regions labelled 3 are blue, the regions labelled 4 are green, the regions labelled 5 are purple, and the regions labelled 6 are orange. Each triangle omits its edges and vertices, and each hexagon omits its two lowest vertices and its rightmost vertex. Show that

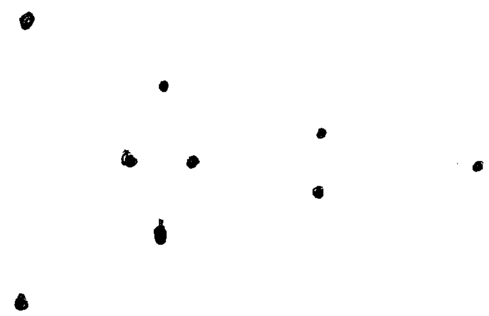
(i) among the points which are colored red, yellow, blue, and green, there do not exist any two points of the same color which are a distance of exactly one unit apart from each other.

(ii) among the points which are colored purple and orange, there do not exist any two points of the same color which are a distance of exactly 2 units apart from each other.



Every hexagon and triangle has all its sides of length 1.

3. (Brawner/Zell) What is special about the set of nine points diagrammed in the figure? Explain, with proof.



4. (Cook/Schaefer) Prove that for any natural number n,

$$\frac{\pi}{4} = \arctan\frac{1}{n} + \arctan\left(\frac{n-1}{n+1}\right).$$

5. (Dizikes/T. Nguyen) It follows from what we learned in class that if a, b, and c are positive integers satisfying $a^2 + b^2 = c^2$, and if a is even and $\gcd(a, b) = 1$, then we can find integers x and y such that $\gcd(x, y) = 1$, x and y are neither both odd nor both even, and

$$a = 2xy, \quad b = x^2 - y^2, \quad \text{and} \quad c = x^2 + y^2.$$

Use this fact (or any other method) to find all the possible pairs of positive integers b and c such that

$$420^2 + b^2 = c^2$$

and $\gcd(420, b) = 1$.

6. (Givens/Tran) Suppose a and n are natural numbers and $\gcd(a, n) = 1$. Prove that there exists a number b such that

$$ab \equiv 1 \pmod{n},$$

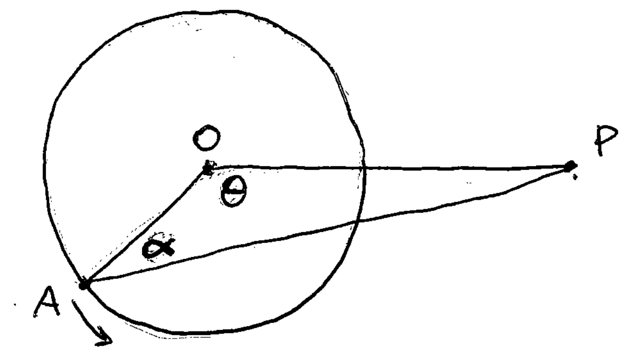
or, in other words, ab leaves a remainder of 1 when divided by n.

7. (Harwell/Lewis) Define

$$T(n) = \begin{cases} \frac{3n+1}{2} & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is even.} \end{cases}$$

Let $T^{(2)}(n) = T(T(n))$, $T^{(3)}(n) = T(T(T(n)))$, and so on. Suppose one could prove that for every natural number n there exists k such that $T^{(k)}(n) < n$. Deduce from this that for every natural number n, there exists k such that $T^{(k)}(n) = 1$.

8. (Harwood/Schwartz) In the diagram, O and P are fixed points, and A is a moving point on a circle of fixed radius r centered at O. Let $\alpha = \angle OAP$ and let $x = AP$. Prove that if $\frac{dx}{dt} = -1$ then the speed of the point A is $\frac{1}{\sin\alpha}$. (Hint: the speed of A is given by $r\frac{d\theta}{dt}$ where $\theta = \angle AOP$. Use the law of cosines and the law of sines.)

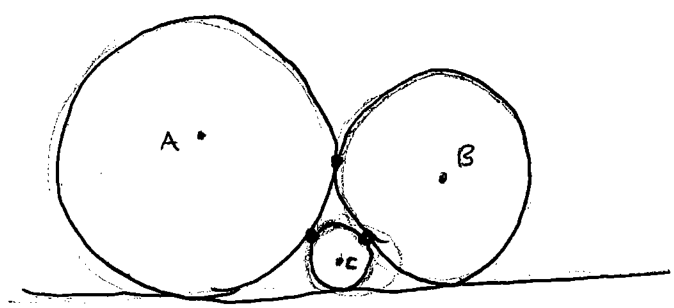


9. (Lerner/Pearce) Let C be a simple closed curve in the plane ("simple" means the curve does not intersect itself). Suppose the origin O is in the region surrounded by C. Prove there exists a line segment AB whose endpoints are on C such that O is the midpoint of AB. (Hint: consider the curve obtained by reflecting the points of C through the origin.)

10. (Matulich/Pitts) Suppose three circles are tangent to each other and to a horizontal line in the configuration shown in the diagram. Let r_1 and r_2 be the radii of the two larger circles and let r_3 be the radius of the smallest circle. Prove that

$$\frac{1}{\sqrt{r_1}} + \frac{1}{\sqrt{r_2}} = \frac{1}{\sqrt{r_3}}.$$

(Hint: draw the triangle ΔABC whose vertices are the centers of the three circles. What are the lengths of the edges of this triangle? Now draw right triangles with each of the sides AB, AC, and BC as hypotenuses, and apply the Pythagorean theorem.)



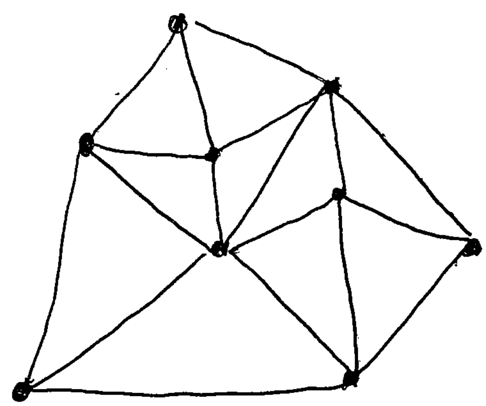
11. (McCord/Perkins) It has been proved that Penrose's kites and darts must tile the plane *aperiodically* if certain matching rules are followed. Show, however, that if we ignore these rules then Penrose's kites and darts can be used to tile the plane *periodically.*

12. (Miller/K. Nguyen) Prove that there exist two irrational numbers p and q such that p^q is rational. (Hint: consider the numbers $\sqrt{2}^{\sqrt{2}}$ and $\left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}}$. Interestingly, you can use these numbers to prove the desired statement without having to decide whether $\sqrt{2}^{\sqrt{2}}$ is rational!)

13. (Monroe/C. Nguyen) Suppose a set of three or more points (called "vertices") is given in the plane, and the vertices are connected by line segments (called "edges") in such a way that all the regions between the edges are triangles (see figure). Let V be the number of vertices, E be the number of edges, and F be the number of triangles. Prove that

$$V - E + F = 1.$$

(Hint: use induction on the number of vertices, starting with $V = 3$.)



(example)

$V = 9$

$E = 18$

$F = 10$