Math 4443/5443
Final Exam

Instructions: Do as many of the problems as you can (total of 100 points).

1. *(15 points)* Suppose $A \subset \mathbf{R}$, and suppose $f : A \to \mathbf{R}$, and (f_n) is a sequence of functions on A. Give definitions of the following statements:

a. f is uniformly continuous on A.
b. (f_n) converges pointwise to f on A.
c. (f_n) converges uniformly to f on A.

2. *(20 points)* Give a proof of the following theorem: "Suppose $f \in \mathcal{R}[a,b]$ and f is continuous at a point $c \in [a,b]$. Then the function F, defined for $x \in [a,b]$ by $F(x) = \int_a^x f$, is differentiable at c, and $F'(c) = f(c)$." (This is part 2 of the Fundamental Theorem of Calculus.)

3. *(10 points)*
a. Give the definition of "Riemann integral of a function on $[a,b]$".
b. Find

$$\lim \frac{1}{n}\left(\frac{1^4}{n^4} + \frac{2^4}{n^4} + \frac{3^4}{n^4} + \cdots + \frac{n^4}{n^4}\right).$$

(Hint: use part **a.**)

4. *(8 points)* Use Taylor's theorem to show that for all $x < 0$,

$$e^x < 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}.$$

5. *(12 points)* Define $f_n : [0,1] \to \mathbf{R}$ by

$$f_n(x) = \frac{nx}{n^2 + x^3}.$$

a. Show that (f_n) converges uniformly on $[0,1]$.
b. Find $\lim \int_0^1 f_n(x)\, dx$. Justify your answer.

6. *(15 points)* Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ is an absolutely convergent series.

a. Show that the series $\sum_{n=1}^{\infty} a_n \sin(nx)$ converges absolutely and uniformly on $\mathbf{R}$.
b. Let $f : \mathbf{R} \to \mathbf{R}$ be defined by

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(nx).$$

Show that f is continuous on $\mathbf{R}$.

7. *(10 points)* Suppose (x_n) is a decreasing sequence with $\lim(x_n) = 0$. Show that the series

$$x_1 - \frac{1}{2}(x_2 + x_3) + \frac{1}{3}(x_4 + x_5 + x_6) - \frac{1}{4}(x_7 + x_8 + x_9 + x_{10}) + \dots$$

converges.

8. *(10 points)* Show that if $|x| < 1$, then

$$\ln(1 - x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}.$$

(Hint: start from the formula for the sum of a geometric series.)