Take-home final
Math 4513

Answer eight of the following ten questions. (Six correct answers are good enough for an A, and three correct are good enough for a B.) Please work independently, and give full explanations for each of your answers. The exam is due by 5 pm on Wednesday, May 7.

1. (Byrnes/Glascock) Show that if you start with a number that is divisible by 3 and perform the reverse-add process, then the result is divisible by 3. (Hint: an easy way to tell if a number is divisible by 3 is to add the digits. If the sum of the number is divisible by 3, then so is the number.)

2. (Millikin/Rainey) Identify the Nash equilibrium for the following 2-player game, in which each player has three strategies. Explain your answer.

$1, 4$	$5, -1$	$0, 1$
$-1, 0$	$-2, -2$	$-3, 4$
$0, 3$	$9, -1$	$5, 0$

3. (Brown/Daifi/Perkins) Suppose a "random Fibonacci sequence" F_n is formed according to the following rule: start with the numbers $F_0 = 1$ and $F_1 = 1$. Now, to determine the value of F_2, flip a coin. If the coin comes up heads, add the values of F_0 and F_1 to get F_2 (thus $F_2 = 2$). If the coin comes up tails, subtract the value of F_1 from F_0 to get F_2 (thus $F_2 = 0$). Now, find the values of F_3, F_4, etc., following the same procedure: for each $n \geq 2$, flip a coin. If the coin comes up heads, take $F_{n+1} = F_{n-1} + F_n$, and if the coin comes up tails, take $F_{n+1} = F_{n-1} - F_n$.

Next, consider the matrices

$$\begin{bmatrix} F_1 \\ F_0 \end{bmatrix}, \begin{bmatrix} F_2 \\ F_1 \end{bmatrix}, \ldots \begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}, \ldots$$

If, when computing F_{n+1}, the coin comes up heads, what 2×2 matrix do you have to multiply $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ by to get $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$?

If, when computing F_{n+1}, the coin comes up tails, what 2×2 matrix do you have to multiply $\begin{bmatrix} F_n \\ F_{n-1} \end{bmatrix}$ by to get $\begin{bmatrix} F_{n+1} \\ F_n \end{bmatrix}$?

If you start off by flipping a tails to get F_2, and then a heads to get F_3, and then a tails to get F_4, what 2×2 matrix do you have to multiply $\begin{bmatrix} F_1 \\ F_0 \end{bmatrix}$ by to get $\begin{bmatrix} F_4 \\ F_3 \end{bmatrix}$? (Show how you got your answer.)

4. (Rees/Thomas) You are playing a game of Nim in which there are four heaps of sizes 11, 13, 7, and 6. (In Nim, each player is allowed to take as many pieces as he wants from any single heap, but cannot take pieces from more than one heap. The player who takes the last piece wins.) You are the first player to move.

What is the Nim sum for the heaps as they stand before you move?

What move can you make to follow the winning strategy? (The winning strategy is to make a move that changes the Nim sum to zero. There is more than one way to do this.)

5. (Lively/Mello de Almeida/Welch) Suppose you are tiling the plane with Penrose kites and darts. There is a process called "inflation" which enables you to start with a tiling containing a small number of kites and darts (say, 1 kite and 1 dart) and continually produce tilings with more and more kites and darts. For this question, we don't need to know the details of how this works. We only need to know the following fact:

If you start with a tiling containing K kites and D darts, and perform the process of inflation n times, then you wind up with a tiling containing K_n kites and D_n darts, where

$$\begin{bmatrix} K_n \\ D_n \end{bmatrix} = P^{2n} \begin{bmatrix} K \\ D \end{bmatrix},$$

and P is the matrix $P = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$.

Use this fact to explain why the ratio of kites to darts (i.e, the ratio of K_n to D_n) approaches the golden section $\phi = \frac{1+\sqrt{5}}{2}$ as n goes to infinity. (Hint: this follows from something we did in class near the beginning of the semester. Think of where you have seen the matrix P before.)

6. (Conrad/Roper) Suppose (p, q) is a pair of twin primes (i.e., p and q are primes and $q = p + 2$). Show that if $p > 3$, then one of the following two statements must be true: either
(i) $p = 12n - 1$ and $q = 12n + 1$ for some natural number n, or
(ii) $p = 12n + 5$ and $q = 12n + 7$ for some natural number n.

7. (Davis/Miller)
(i) Show that if n is an even number, then n^2 is even; and if n is an odd number, then n^2 is odd.
(ii) Show that if a, b, c is a primitive Pythagorean triple, then exactly one of the two numbers a and b is even. (That is, a and b cannot both be odd, and cannot both be even. Hint: use Euclid's formula for primitive Pythagorean triples.)

8. (Anderson/Smith) Find a set of five homogeneous polynomials of degree 2 in the variables x, y, and z such that both the following are true:

(i) each of these five polynomials in the set is a solution of Laplace's equation.
(ii) every homogeneous polynomial of degree 2 which solves Laplace's equation is a linear combination of these five polynomials.

(Statement (ii) means that, if P_1, P_2, P_3, P_4, P_5 are the five polynomials, then every homogeneous polynomial of degree 2 which solves Laplace's equation is of the form

$$a_1P_1 + a_2P_2 + a_3P_3 + a_4P_4 + a_5P_5,$$

where a_1, a_2, a_3, a_4, a_5 are constants. Finding the polynomials boils down to a simple exercise in linear algebra. You can probably guess a correct set of five polynomials without much trouble, but you should also explain how you know that your set satisfies (ii).)

9. (Morris/Vance) Consider again the "random Fibonacci sequence" defined in Problem 3 above. Is it a Markov chain? Explain your answer.

(Here is what is involved in answering this question. Suppose you are looking at a certain value of n, and you know what the value of F_n is, say $F_n = 7$ for example. Given this knowledge, there is then a certain probability P that F_{n+1} will take, say the value 9. If the sequence is a Markov chain, then this probability P will not depend on what any of the values F_1 through F_{n-1} are; i.e., knowing the values of F_1 through F_{n-1} will not give you any extra information about the probability that F_{n+1} equals 9. On the other hand, if knowing one or more of the values F_1 through F_{n-1} (in addition to knowing that $F_n = 7$) does give you more information about the probability that $F_{n+1} = 9$, then the sequence is not a Markov chain.)

10. How many ways can each of 5 persons be given a right glove and a left glove from 6 distinguishable pairs of gloves, but with no person getting two gloves from the same pair?