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Math 4163
Final Exam

Name: Answer key

NOTE: On this exam, you may use any solution formula from class, without having to rederive it.

1. *(20 points)* Consider the heat equation with source term:

$$\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + \alpha u,$$

for $0 < x < L$ and $t > 0$. Suppose α is a constant and $\alpha < 0$. Use separation of variables to find the solution that satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(L,t) = 0 \qquad (\text{for } t \geq 0)$$

and the initial condition

$$u(x,0) = f(x) \qquad (\text{for } 0 \leq x \leq L).$$

Remember to give a formula for the coefficients in your series solution in terms of f.

Putting $u = G(t)\cdot\phi(x)$, we get (2) $\frac{dG}{dt}\cdot\phi = kG\frac{d^2\phi}{dx^2} + \alpha G\phi$, and dividing by $G\phi$ gives $\frac{\left(\frac{dG}{dt}\right)}{G} = \frac{k\left(\frac{d^2\phi}{dx^2}\right)}{\phi} + \alpha$ (2) $= -\lambda$ (= constant, since the variables are separated.)

So $\frac{dG}{dt} = -\lambda G$ (2) and $\frac{d^2\phi}{dx^2} = \frac{-\lambda-\alpha}{k}\cdot\phi = -\left(\frac{\lambda+\alpha}{k}\right)\cdot\phi$. (2) Since $\phi(0)=\phi(L)=0$, the non-trivial solutions to the latter equation are $\phi(x) = \sin\left(\frac{n\pi x}{L}\right)$ (2), where $n = 1, 2, 3, \ldots$ and $\left(\frac{\lambda+\alpha}{k}\right) = \left(\frac{n^2\pi^2}{L^2}\right)$. (2) Then $\lambda = \frac{kn^2\pi^2}{L^2} - \alpha$ (1), so $G(t) = Ce^{-\lambda t} =$

$= Ce^{\left(\alpha - \frac{kn^2\pi^2}{L^2}\right)t}$ (2). Thus the series solution for u is

$$\boxed{u = \sum_{n=1}^{\infty} C_n e^{\left(\alpha - \frac{kn^2\pi^2}{L^2}\right)t}\sin\left(\frac{n\pi x}{L}\right).}$$ (2)

Putting $t=0$ gives

$f(x) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi x}{L}\right)$, so $\boxed{C_n = \frac{2}{L}\int_0^L f(w)\sin\left(\frac{n\pi w}{L}\right)dw}$ (3)

2. *(14 points)* Consider the heat equation with source term:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 9u,$$

for $0 < x < \pi$ and $t > 0$. Suppose u satisfies the boundary conditions

$$u(0,t) = 0 \quad \text{and} \quad u(\pi,t) = 0 \qquad (\text{for } t \geq 0).$$

[8] **a.** This problem has non-zero equilibrium solutions. Find them. 2

For an equilibrium, $\frac{\partial u}{\partial t} = 0$ 2, so $\frac{\partial^2 u}{\partial x^2} + 9u = 0$, or $\frac{d^2 u}{dx^2} = -9u$.

The general solution to this ODE is ~~u~~ $u = A\cos 3x + B\sin 3x$. 2

But $u(0) = 0 \Rightarrow 0 = A$, so $u = B\sin 3x$. 2 Also, $u(\pi) = 0$ is satisfied for any B, since $\sin 3\pi = 0$. So $\boxed{u = B\sin 3x \text{ is an equilibrium solution, for any } B.}$

b. Graph one of the solutions you found in part **a**, and for the solution you graphed, answer the following questions (with a brief explanation):

2 (i) is heat flowing to the right or to the left at $x = 0$?

Since $\frac{\partial u}{\partial x} > 0$ at $x = 0$, and by Fourier's law $\phi = -k\frac{\partial u}{\partial x} = -\frac{\partial u}{\partial x}$, then the heat flux ϕ is negative at $x = 0$, so heat is flowing to the left.

2 (ii) is heat flowing to the right or to the left at $x = \pi$?

Since $\frac{\partial u}{\partial x} < 0$ at $x = \pi$, then $\phi = -\frac{\partial u}{\partial x}$ is positive at $x = \pi$, so heat is flowing to the right.

(Note: taking a negative value of B, say $u = -\sin 3x$, would give a solution for which the answers to (i) and (ii) are reversed.)

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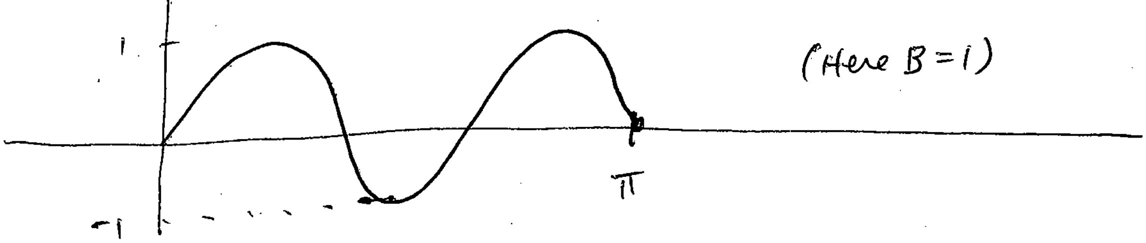
$u(x) = \sin 3x$



(Here $B = 1$)

3. *(20 points)* Consider the function $f(\theta)$ defined for $-\pi < \theta \le \pi$ by

$$f(\theta) = \begin{cases} 1 & \text{if } 0 \le \theta \le \pi \\ 0 & \text{if } \pi < \theta < 2\pi \rightarrow -\pi < \theta < 0. \end{cases}$$

[10] **a.** Find the coefficients of the Fourier series for f on $[-\pi, \pi]$. Write out the first three non-zero terms of the series.

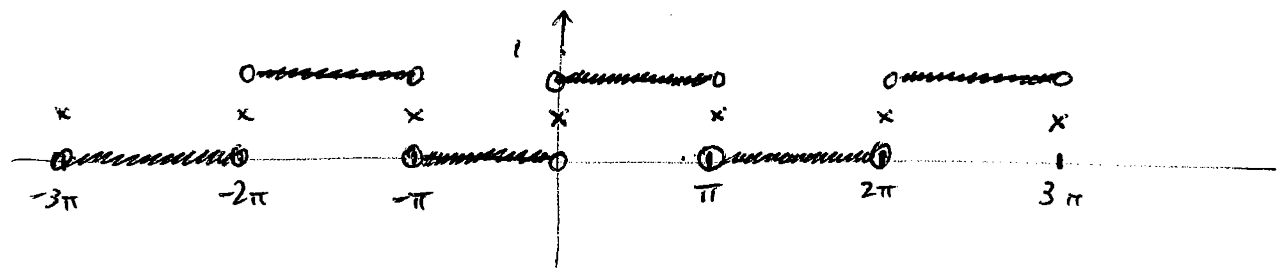
$$f(\theta) = A_0 + \sum_{n=1}^{\infty} A_n \cos n\theta + B_n \sin n\theta \quad 2 \quad \text{where } A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{\pi} 1\,d\theta =$$

$$= \frac{1}{2\pi}\cdot\pi = \frac{1}{2} \quad 2, \text{ and (for } n \ge 1) \; A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi}\cos(n\theta)\,d\theta = \frac{1}{\pi}\left[\sin n\pi - \sin 0\right] = 0 \quad 2;$$

$$\text{and } B_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta = \frac{1}{\pi}\int_0^{\pi}\sin(n\theta)\,d\theta = \frac{1}{\pi}\left[\frac{-\cos(n\pi) + \cos 0}{n}\right] = \frac{(1-(-1)^n)}{n\pi} \quad 2.$$

$$\text{So } \boxed{f(\theta) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n\pi}\sin n\theta = \frac{1}{2} + \frac{2}{\pi}\sin\theta + \frac{2}{3\pi}\sin 3\theta + \dots} \quad 2$$

[4] **b.** Sketch the graph of the function to which the Fourier series converges on $[-3\pi, 3\pi]$. Use an $\times$ to mark the values of the function at the points where it is discontinuous.



2 period 2π

2 correct values at discontinuities

[4] **c.** Find, in series form, the solution $u(r, \theta)$ of Laplace's equation $\nabla^2 u = 0$ in the circle $0 \le r \le 1$, $-\pi < \theta \le \pi$ that satisfies the boundary condition $u(1, \theta) = f(\theta)$. (You do not need to rederive the series form for the solution of Laplace's equation on the circle.) Write out the first three non-zero terms of the series.

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n (A_n \cos n\theta + B_n \sin n\theta) \quad 2$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} r^n \frac{(1-(-1)^n)}{n\pi}\sin n\theta = \frac{1}{2} + \frac{2r}{\pi}\sin\theta + \frac{2r^3}{3\pi}\sin 3\theta + \dots \quad 2$$

[2] **b.** Give the value of u at $r = 0$.

$$u(0,\theta) = \frac{1}{2} + 0 + 0 + \dots = \frac{1}{2}. \quad 2$$

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4. *(22 points)* Consider the wave equation for an elastic string,

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

for $0 < x < L$ and $t > 0$.

[18] **a.** Use separation of variables to find the solution which satisfies the boundary conditions

$$\frac{\partial u}{\partial x}(0,t) = 0 \quad \text{and} \quad \frac{\partial u}{\partial x}(L,t) = 0 \qquad (\text{for } t \geq 0)$$

and the initial conditions

$$u(x,0) = f(x) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) = 0 \qquad (\text{for } 0 \leq x \leq L).$$

Remember to give formulas for the coefficients in your series solution in terms of f.

Putting $u = G(t)\cdot\phi(x)$, we get 1 $\frac{d^2G}{dt^2}\cdot\phi = c^2 G\cdot\frac{d^2\phi}{dx^2}$, and dividing by $(G\phi c^2)$ gives $\dfrac{\frac{d^2G}{dt^2}}{c^2 G} = \dfrac{\frac{d^2\phi}{dx^2}}{\phi}$ 2 $= -\lambda$, Since $\frac{d\phi}{dx}(0) = \frac{d\phi}{dx}(L) = 0$,

The solutions of the ODE $\frac{d^2\phi}{dx^2} = -\lambda\phi$ are $\phi(x) =$ 2 $\cos\left(\frac{n\pi x}{L}\right)$ for $n = 0, 1, 2, \ldots$, with corresponding values of $\lambda = \frac{n^2\pi^2}{L^2}$ 2. Then the ODE for G is $\frac{d^2G}{dt^2} = \frac{-c^2n^2\pi^2}{L^2}G$, 2 which has general solution (*) $G(t) = A\cos\left(\frac{n\pi ct}{L}\right) +$ 2 $B\sin\left(\frac{n\pi ct}{L}\right)$. So the series solution is 2 $u(x,t) = \sum_{n=0}^{\infty}\left\{A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right\}\cos\left(\frac{n\pi x}{L}\right)$.

Putting $t=0$ gives 2 $f(x) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi x}{L}\right)$, so 2 $\boxed{\begin{aligned} A_0 &= \frac{1}{L}\int_0^L f(w)\,dw, \\ A_n &= \frac{2}{L}\int_0^L f(w)\cos\left(\frac{n\pi w}{L}\right)dw \quad (n\geq 1)\end{aligned}}$

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Also, $\frac{du}{dt} = \sum_{n=0}^{\infty}\left(\frac{n\pi c}{L}\right)\left\{-A_n\sin\left(\frac{n\pi ct}{L}\right) + B_n\cos\left(\frac{n\pi ct}{L}\right)\right\}\cos\left(\frac{n\pi x}{L}\right)$, and putting $t=0$ gives $0 = \sum_{n=0}^{\infty}\left(\frac{n\pi c}{L}\right)B_n\cos\left(\frac{n\pi x}{L}\right)$, so 2 $B_n = 0$. Thus $\boxed{u(x,t) = \sum_{n=0}^{\infty} A_n\cos\left(\frac{n\pi ct}{L}\right)\cos\left(\frac{n\pi x}{L}\right)}$ where A_n are as above.

[4] **b.** Give the explicit solution in the case when $f(x) = 1$ for $0 < x < L$. How would you describe the motion of the string which this solution represents?

When $f(x) = 1$ we have $A_0 = \frac{1}{L}\int_0^L 1\,dw = \frac{1}{L}\cdot L = 1$, and for $n \geq 1$,

$A_n = \frac{2}{L}\int_0^L 1\cdot\cos\left(\frac{n\pi w}{L}\right)dw = \frac{2}{n\pi}(\sin n\pi - \sin 0) = 0$. So

$\boxed{u(x,t) = 1}$. The string does not move, it stays flat and horizontal at all times

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(*) If $n \neq 0$. If $n = 0$, then $G(t) = A + Bt$. But $\frac{dG}{dt} = 0$ at $t = 0 \Rightarrow B = 0$.

5. *(12 points)* Suppose f is piecewise smooth on $[0, L]$ and $A_n = \int_0^L f(w) \cos \frac{n\pi w}{L}\, dw$. Show that

$$\int_0^L f'(w) \sin \frac{n\pi w}{L}\, dw = -\left(\frac{n\pi}{L}\right) A_n.$$

(Hint: use integration by parts.)

Take $u = \sin \frac{n\pi w}{L}$, $dv = f'(w)\, dw$

Then $du = \left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi w}{L}\right) dw$, $v = f(w)$.

So $\int_0^L f'(w) \sin \frac{n\pi w}{L}\, dw = \int_0^L u\, dv = \left[uv\right]_0^L - \int_0^L v\, du =$

$= \left[\sin\left(\frac{n\pi w}{L}\right) f'(w)\right]_{w=0}^{w=L} - \int_0^L f(w) \left(\frac{n\pi}{L}\right) \cos\left(\frac{n\pi w}{L}\right) dw =$

$= \sin(n\pi) f'(L) - \sin 0 \cdot f'(0) - \left(\frac{n\pi}{L}\right) \int_0^L f(w) \cos \frac{n\pi w}{L}\, dw =$

$= 0 - 0 - \left(\frac{n\pi}{L}\right) A_n = -\left(\frac{n\pi}{L}\right) A_n.$

6. *(12 points)* Suppose u solves $\frac{\partial u}{\partial t} = \nabla^2 u + 3$ on the rectangle R given by $0 < x < 2$, $0 < y < 5$, and the normal derivative $\nabla u \cdot \mathbf{n}$ is zero at all points on the boundary, for all $t > 0$. Find (with explanation) $\frac{d}{dt} \int \int_R u\, dx\, dy$.

$$\frac{d}{dt} \iint_R u\, dx\, dy = \iint_R \frac{\partial u}{\partial t}\, dx\, dy = \iint_R \left(\nabla^2 u + 3\right) dx\, dy =$$

$$= \iint_R \nabla^2 u\, dx\, dy + \iint_R 3\, dx\, dy.$$

By the divergence theorem, or just by carrying out the integration, we have that $\iint_R \nabla^2 u\, dx\, dy$ equals the integral over the boundary of R of $\nabla u \cdot \vec{n}$. So $\iint_R \nabla^2 u\, dx\, dy = 0$, and

$$\frac{d}{dt} \iint_R u\, dx\, dy = \iint_R 3\, dx\, dy = 3 \times (\text{area of } R) = 3 \cdot 2 \cdot 5 = 30.$$