Solutions to Problems on last assignment

9.3.5 (First proof) The given series is of the form $\sum x_n y_n$, where

$$(y_n) = (1, -1, -1, 1, 1, 1, -1, -1, 1, 1, \dots)$$

and $x_n = 1/n$. The partial sums of $\sum y_n$ are given by the bounded sequence $(s_n) = (1, 0, -1, 0, 1, 0, -1, 0, 1, \dots)$, and x_n is a decreasing sequence with limit 0, so by Dirichlet's Test (9.3.4), $\sum x_n y_n$ converges.

(Second proof) Alternatively, we could look at the series

$$1 - (\frac{1}{2} + \frac{1}{3}) + (\frac{1}{4} + \frac{1}{5}) - (\frac{1}{6} + \frac{1}{7}) - (\frac{1}{8} + \frac{1}{9}) + \cdots = 1 + \sum_{n=1}^{\infty} (-1)^n \left(\frac{1}{2n} + \frac{1}{2n+1} \right).$$

The convergence of this series follows from the Alternating Series Test (9.3.2). Note, however, that it doesn't follow immediately that the series

$$\sum a_n = 1 - \frac{1}{2} - \frac{1}{3} + \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \frac{1}{7} + \dots$$

converges: the two series are not the same, since parentheses were introduced into one to get the other. Theorem 9.1.3 cannot be applied here, since it assumes that the series in question is known to be convergent, and we don't know yet that $\sum a_n$ is convergent.

Instead, let (t_n) be the sequence of partial sums of $\sum a_n$. It's easy to see that (t_{2k}) is the sequence of partial sums of an alternating series, which converges by the Alternating Series test. A similar argument also shows that (t_{2k+1}) converges. Since $\lim(a_k) = 0$ and $t_{2k+1} - t_{2k} = a_{2k+1}$, then (t_{2k}) and (t_{2k+1}) must converge to the same limit. The convergence of (t_n) then follows from exercise 5 at the end of section 3.4. Hence $\sum a_n$ converges.

9.3.10 By Abel's Lemma with $n = 0$,

$$\begin{aligned}
\sum_{k=1}^{m} a_k \cdot \frac{1}{k} &= s_m \frac{1}{m} + \sum_{k=1}^{m-1} \left(\frac{1}{k} - \frac{1}{k+1} \right) s_k \\
&= \frac{s_m}{m} + \sum_{k=1}^{m-1} \left(\frac{1}{k^2 + k} \right) s_k
\end{aligned}$$

Let M be such that $|s_k| \leq M$ for all k. Then the latter series in the above equation converges absolutely, by comparison to $\sum \frac{M}{k^2}$. Also, $\lim(s_m/m) = 0$ by exercise 7 in section 3.2. Hence, if we take the limit with respect to m in the above equation, it follows from Theorem 3.2.3 and the above equation that

$$\sum_{k=1}^{\infty} \frac{a_k}{k} = 0 + \sum_{k=1}^{\infty} \left(\frac{1}{k^2 + k} \right) s_k.$$

9.4.6

a) $\rho = \lim((1/n^n)^{1/n}) = \lim(1/n) = 0$, so the radius of convergence is ∞.

b) $\lim a_n/a_{n+1} = \lim\left(\frac{n}{n+1}\right)^{\alpha}(n+1)$. Since

$$\lim\left(\frac{n}{n+1}\right)^{\alpha} = \lim\left(\frac{1}{1+1/n}\right)^{\alpha} = \lim\left(\frac{n}{n+1}\right)^{\alpha} = 1^{\alpha} = 1,$$

and $\lim(n+1) = \infty$, it follows that $\lim a_n/a_{n+1} = \infty$, so $R = \infty$.

c)

$$R = \lim \frac{a_n}{a_{n+1}} = \lim\left(\frac{1}{(1+1/n)^n}\right) = 1/e$$

d)

$$R = \lim \frac{a_n}{a_{n+1}} = \lim \frac{\ln(n+1)}{\ln n} = \lim \frac{1/(n+1)}{1/n} = \lim n/(n+1) = 1.$$

e)

$$R = \lim \frac{a_n}{a_{n+1}} = \lim \frac{(2n+1)(2n+2)}{(n+1)^2} = 4.$$

f)

$$\log \lim(n^{1/\sqrt{n}}) = \lim \log(n^{1/\sqrt{n}}) = \lim \frac{\log n}{\sqrt{n}} = 0 \quad \text{(by L'hopital's rule)},$$

so

$$\lim(n^{1/\sqrt{n}}) = e^0 = 1.$$

Hence

$$\rho = \lim \frac{1}{n^{1/\sqrt{n}}} = 1,$$

so $R = 1$.

9.4.17 We know that for $|x| < 1$,

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n,$$

and hence, replacing x by $-x^2$, we get

$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-1)^n x^{2n}.$$

Integrating both sides from 0 to x, and using Theorem 9.4.11, we get that

$$\arctan x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$$

for $|x| < 1$.