[20] Using the Euclidean algorithm:

1.

43	23		
23	20 2	$43 - 23 = 20$	2
20	3 2	$23 - 20 = 3$ $23 - (43-23) = 3$ $2\cdot 23 - 43 = 3$	3
3	2 2	$20 - 6\cdot 3 = 2$ $(43-23) - 6(2\cdot 23 - 43) = 2$ $7\cdot 43 - 13\cdot 23 = 2$	3
2	1 2	$3 - 2 = 1$ $(2\cdot 23 - 43) - (7\cdot 43 - 13\cdot 23) = 1$ $15\cdot 23 - 8\cdot 43 = 1$	3

So we can take $x = -8$ and $y = 15$. 1

---

[20]

2. By the prime factorization theorem, we can write n
5 as a product of primes:

$$n = p_1 \cdot p_2 \cdot p_3 \cdots p_m \quad ; \text{where } p_1, p_2, \ldots, p_m \text{ are primes.}$$

Since n is of the form $3k+2$, then n is not divisible by 3,
2 so none of the primes $p_1, p_2, \ldots, p_m$ is equal to 3 (or divisible by 3).
Therefore every one of the primes $p_1, p_2, \ldots, p_m$ leaves a remainder
2 of either 1 or 2 when divided by 3; i.e. each prime is either of the form $3k+1$ or of the form $3k+2$.

We want to show that at least one of the primes
3 is of the form $3k+2$. To prove this we can assume that all the primes are of the form $3k+1$, and derive a contradiction.

3 But the product of any two numbers of the form $3k+1$ is also of the form $3k+1$: $(3k+1)(3r+1) = 9kr + 3k + 3r + 1 = 3(3kr + k + r) + 1$.

5 So if all the primes $p_1, \ldots, p_m$ are of the form $3k+1$, then so is their product, which equals n. This is a contradiction since n is of the form $3k+2$.

[15]

3. (i) (2) - First construct a line L_1 perpendicular to AC at P.

(2) - Find the intersection of L_1 with side BC, call it Q

(3) - Construct a circle with center at P and radius PQ, call it $\mathcal{C}$.

(2) - Find the intersection of $\mathcal{C}$ with AC, call it S.

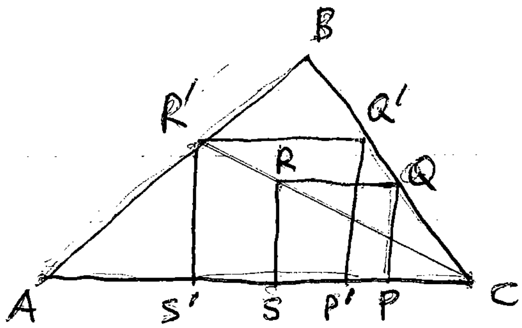
(2) - Construct a perpendicular line L_2 to AC at S

(2) - Construct a perpendicular line L_3 to PQ at Q.

(2) - Find the intersection of L_2 and L_3, call it R.

[15]

(ii)



From the diagram, we see that by similar triangles,

(3) $$\frac{R'Q'}{RQ} = \frac{R'C}{RC} = \frac{R'S'}{RS}.$$

Hence $\dfrac{R'Q'}{R'S'} = \dfrac{RQ}{RS} = 1$. So $R'Q'S'P'$ is a square.

This suggests the following construction:

(2) - First construct a square PQRS as in part (i). (It doesn't matter which point P we use to start with.)

(2) - Draw line CR through C and R.

(2) - Let R' be the intersection of line CR with side AB.

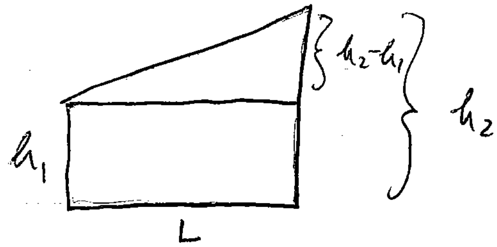
(2) - Draw the line through R' perpendicular to AC; let S' be the intersection of this line with AC.

(2) - Draw the circle ~~thro~~ with center R' and radius R'S'; let Q' be the intersection of this circle with side BC.

(2) - Draw the line through Q' perpendicular to AC; let P' be the intersection of this line with AC.

[10]

4. (i) Cutting the trapezoid into a triangle and rectangle as shown, 3



we get

(Area trapezoid) = (Area rectangle) + (Area triangle)

$$= h_1 \cdot L \; ② + \tfrac{1}{2}(L)(h_2-h_1) \; ②$$

$$= h_1 L + \frac{Lh_2}{2} - \frac{Lh_1}{2} = \frac{Lh_1}{2} + \frac{Lh_2}{2} = \frac{1}{2}(h_1+h_2)L \; ③$$

[20]

(ii) From part (i) we know the area of the trapezoid in Fig. 5 is $\boxed{A = \frac{1}{2}(a+b)\cdot(a+b)}$. 5

But the area is also given by

$$A = (\text{area triangle ①}) + (\text{area triangle ②}) + (\text{area triangle ③})$$

or $$A = \frac{1}{2}ab + \frac{1}{2}c\cdot c + \frac{1}{2}a\cdot b$$

or $$\boxed{A = ab + \frac{c^2}{2}}$$ 5

Combining these two formulas for A gives

$$\frac{1}{2}(a+b)(a+b) = ab + \frac{c^2}{2}$$

$$\Rightarrow \frac{1}{2}(a^2+b^2+2ab) = ab + \frac{c^2}{2}$$ 10

$$\Rightarrow \frac{1}{2}(a^2+b^2) + ab = ab + \frac{c^2}{2}$$

$$\Rightarrow \frac{1}{2}(a^2+b^2) = \frac{c^2}{2}$$

$$\Rightarrow a^2+b^2 = c^2$$, as desired.