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Math 4513
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April 29, 2008

Risk and Markov Chains

A *Markov chain* is characterized as follows: We have a set of *states*, S = {s1, s2, …,sr}. The process starts in one of these states and moves successively from one state to another. Each move is called a *step.* If the chain is currently in state *si*, then it moves to state *sj* at the next step with a probability denoted by *pij*, and this probability does not depend upon which states the chain was in before the current state. [5]

$$\Pr(X_{n+1} = x | X_n = x_n, \ldots, X_1 = x_1) = \Pr(X_{n+1} = x | X_n = x_n).$$

Land of Oz example:

Land of Oz example demonstrates the following "theorem": Let P be the transition matrix of a Markov chain. The ijth entry pij^n of the matrix P^n gives the probability that the Markov chain, starting in state si, will be in state sj after n steps. (see page at end of report for matrices corresponding to these examples)

Absorbing Markov chains relate to Risk:

Definition: A state si of a Markov chain is called absorbing if it is impossible to leave. A Markov chain is absorbing if it has at least one absorbing state, and if from every state it is possible to reach an absorbing state (not necessarily in one step). A state which is not absorbing is called transient (Chapter 11).

Drunkard Walk Example:

Canonical form

Theorem:

In an absorbing Markov chain, the probability that the process will be absorbed is 1 (i.e., Q^n→0 as n → ∞)

Proof: From each non-absorbing state sj it is possible to reach an absorbing state. Let mj be the minimum number of steps required to reach an absorbing state, starting from sj. Let pj be the probability that, starting from sj, the process will not reach an absorbing state in mj steps. Then pj< 1. Let m be the largest of the mj and let p be the largest of the pj. The probability of not being absorbed in m steps is less than or equal to p, in 2m steps less than or equal to p^2, etc. Since p <1 these probabilities tend to 0. Since the probability of not being absorbed in n steps is monotone decreasing, the probabilities also tend to 0, hence lim as n →∞, Q^n = 0.

Terminology:

According to the official rules of Risk, an <u>invasion</u> is the entire attack of a country and a <u>battle</u> is an individual role of the dice. [3]

Single Victor:

Because all battles of Risk can be modeled using Absorbing Markov Chains, all battles must end. In addition, because there is no absorbing state in which neither army has units left, there must be one and only one victor in any given battle. This same logic can be applied to a Risk game as a whole to conclude that all games of Risk must end, and in each game, there must be one and only one victor.

Throughout the following:
A = size of attacker's invading army
a = current size of attacker's army
D = size of defender's army
d = current size of defender's army

Sides of dice:
The number of sides on a dice, s, has a great affect on the outcome of an invasion. The greater the number of sides = the greater the advantage is to the attacker. [1] This happens because it decreases the probability of the defender rolling the highest number (which is unbeatable by the attacker).

For example, consider a one army versus one army invasion (A = 1, D = 1). If coins (s = 2), were used, there could only be 4 outcomes. Let H > T:

(H, H), (H, T), (T, H), (T, T)

Because the defender wins in the event of a tie:

P(Attacker winning | A = 1, D = 1, s = 2) = 1/4 = .25.

But if dice were used, there would be 36 possible outcomes (6*6 = 36 possible outcomes).

The number of outcomes in which the attacker wins = 5 + 4 + 3 + 2 + 1 + 0 = 15.

P(Attacker winning | A = 1, D = 1,s = 6) = 15/36 = .4167 > .25 = P(Attacker winning | s = 2)

(See Table 1 for general formulas for different s)

P(Attacker winning) | A = a, D = d, s = 6):
This can be modeled using Absorbing Markov Chains in Canonical form.

Let $\mathbf{S} = (\mathbf{I} - \mathbf{Q})^{-1}\mathbf{R}$ [1].

To determine P(Attacker winning), look at row (A, D) of matrix S. Add up the intersections of the rows were the attacker wins. [1]

Example 2 (see example 2 for matrices):
(Note: In this example, Blatt uses the size of the army in the attacker's country of origin, not the size of the attacking army)

Let A = 4 and D = 3

P(Attacker winning | s = 6, a = 4, d = 3) = 0.0754 + 0.1496 + 0.2452 = 0.4702 [1]

This means there is a 47.02% chance that the attacker will end up winning this battle. [1]
(There is dispute amongst the papers as to the actual value so I went with the one that fits this example)
(See Table 2 for P(Attacker winning | s = 6, A < 10, D < 10)

P(Attacker winning | s = 6, A = D > 5) > .5 [2]

(See Figure 1 for graphical representation)

Conclusions:

If the attacker and defender have the same number of units greater than 5, the advantage is to the attacker. This advantage continues to increase as the numbers increase. Therefore, a general strategy is to continue to build up troops along borders, but attack before the other player does.
Further research would need to be done to further develop a strategy at Risk: find the number of dice that most closely resembles a 50-50 probability for both attacker and defender at even strength as numbers get higher; find how the probabilities change when rolling for all troops at once (each dice is half of army for defender, each dice is third of army for attacker); find the affect of assigning territories versus selecting them (old version randomly assigned them through the cards); find which continents are most advantageous to conquer; and find how reinforcements affect the outcome (from territories, continents, and cards) . From this a more general strategy can be made for an entire game of Risk.

- "In the typical case -- attacker rolling three dice, defender rolling two -- the defender is likely to lose six armies for every five lost by the attacker. Keep that in mind." – Hasbro [4]

- "When you're trying to figure out how much territory your army can conquer (or how many armies you need to take a chunk of territories), you can figure it out simply by using this formula:

  Enemy armies to be defeated
  \+ Number of territories to be occupied
  = Armies needed" – Hasbro [4]

Table 1: General Formulas with varying *s*

6 sided dice					Transition	General Formula
Case	a	d	From state	To state	Probability	($s = \#$ of faces on die)
I	2	1	(2,1)	(2,0)	0.4166	$\frac{s-1}{2s}$
				(0,1)	0.5834	$\frac{s+1}{2s}$
II	3	1	(3,1)	(3,0)	0.5787	$\frac{(s-1)(4s+1)}{6s^2}$
				(2,1)	0.4213	$\frac{(s+1)(2s+1)}{6s^2}$
III	≥ 4	1	(a,1)	(a,0)	0.6597	$\frac{(s-1)(3s+1)}{4s^2}$
				(a-1,1)	0.3403	$\frac{(s+1)^2}{4s^2}$
IV	2	≥ 2	(2,d)	(2,d-1)	0.2546	$\frac{(s-1)(2s-1)}{6s^2}$
				(1,d)	0.7454	$\frac{(s+1)(4s-1)}{6s^2}$
V	3	≥ 2	(3,d)	(3,d-2)	0.2276	$\frac{(s-1)(2s^2-2s-1)}{6s^3}$
				(1,d)	0.4483	$\frac{(s+1)(2s^2+2s-1)}{6s^3}$
				(2,d-1)	0.3241	$\frac{(s-1)(s+1)}{3s^2}$
VI	≥ 4	≥ 2	(a,d)	(a,d-2)	0.3717	$\frac{(s-1)(6s^3-3s^2-5s-2)}{12s^4}$
				(a-2,d)	0.2926	$\frac{(s+1)(2s+1)(3s^2+3s-1)}{30s^4}$
				(a-1,d-1)	0.3357	$\frac{(s+1)(s-1)(18s^2+15s+8)}{60s^4}$

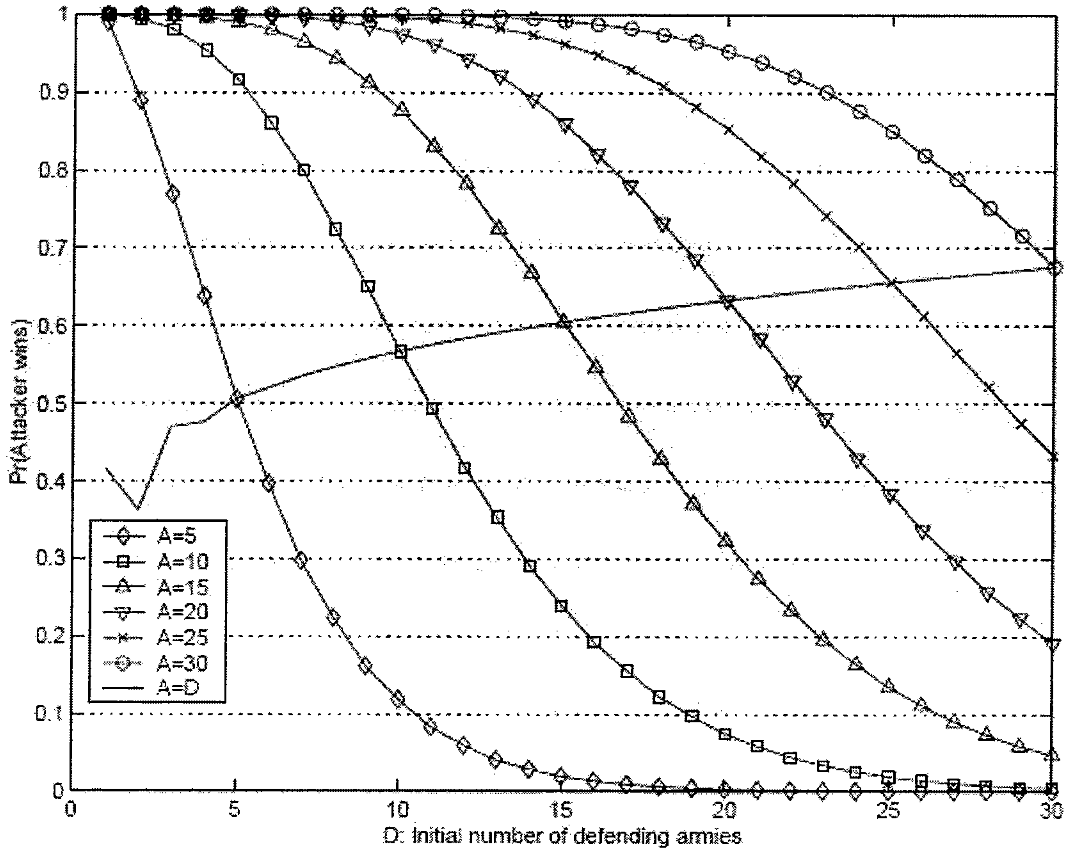
[1]

Table 2: Probability that the attacker wins (A = A-1)

A \ D	1	2	3	4	5	6	7	8	9	10
1	0.417	0.106	0.027	0.007	0.002	0.000	0.000	0.000	0.000	0.000
2	0.754	0.363	0.206	0.091	0.049	0.021	0.011	0.005	0.003	0.001
3	0.916	0.656	0.470	0.315	0.206	0.134	0.084	0.054	0.033	0.021
4	0.972	0.785	0.642	0.477	0.359	0.253	0.181	0.123	0.086	0.057
5	0.990	0.890	0.769	0.638	0.506	0.397	0.297	0.224	0.162	0.118
6	0.997	0.934	0.857	0.745	0.638	0.521	0.423	0.329	0.258	0.193
7	0.999	0.967	0.910	0.834	0.736	0.640	0.536	0.446	0.357	0.287
8	1.000	0.980	0.947	0.888	0.818	0.730	0.643	0.547	0.464	0.380
9	1.000	0.990	0.967	0.930	0.873	0.808	0.726	0.646	0.558	0.480
10	1.000	0.994	0.981	0.954	0.916	0.861	0.800	0.724	0.650	0.568

[2]

Figure 1: Attacker's winning probabilities at various strengths



[2]

Example 2: Let A +1 = 4 and D = 3.

(Note: In this example, Blatt uses the size of the army in the attacker's country of origin, not the size of the attacking army)

	(2,1)	(2,2)	(2,3)	(3,1)	(3,2)	(3,3)	(4,1)	(4,2)	(4,3)
(2,1)	0	0	0	0	0	0	0	0	0
(2,2)	0.2546	0	0	0	0	0	0	0	0
(2,3)	0	0.2546	0	0	0	0	0	0	0
(3,1)	0.4213	0	0	0	0	0	0	0	0
(3,2)	0.3241	0	0	0	0	0	0	0	0
(3,3)	0	0.3241	0	0.2276	0	0	0	0	0
(4,1)	0	0	0	0.3403	0	0	0	0	0
(4,2)	0	0.2926	0	0.3357	0	0	0	0	0
(4,3)	0	0	0.2926	0	0.3357	0	0.3717	0	0

Matrix Q

	(2,0)	(3,0)	(4,0)	(1,1)	(1,2)	(1,3)
(2,1)	0.4166	0	0	0.5834	0	0
(2,2)	0	0	0	0	0.7454	0
(2,3)	0	0	0	0	0	0.7454
(3,1)	0	0.5787	0	0	0	0
(3,2)	0	0.2276	0	0	0.4483	0
(3,3)	0	0	0	0	0	0.4483
(4,1)	0	0	0.6597	0	0	0
(4,2)	0	0	0.3717	0	0	0
(4,3)	0	0	0	0	0	0

Matrix R

[1]

Matrix $S = (I - Q)^{-1} R =$

	(2,0)	(3,0)	(4,0)	(1,1)	(1,2)	(1,3)
(2,1)	0.4166	0	0	0.5834	0	0
(2,2)	0.1061	0	0	0.1485	0.7454	0
(2,3)	0.0270	0	0	0.0378	0.1898	0.7454
(3,1)	0.1755	0.5787	0	0.2458	0	0
(3,2)	0.1350	0.2276	0	0.1891	0.4483	0
(3,3)	0.0743	0.1317	0	0.1041	0.2416	0.4483
(4,1)	0.0597	0.1969	0.6597	0.0836	0	0
(4,2)	0.0899	0.1943	0.3717	0.1260	0.2181	0
(4,3)	0.0754	0.1496	0.2452	0.1056	0.2060	0.2181

Matrix S

[1]

Works Cited

[1] Blatt, Sharon, *RISKy business: An in-depth look at the game RISK Undergraduate Math Journal*, Vol. 3, No. 2, 2002, http://www.rose-hulman.edu/mathjournal/archives/2002/vol3-n2/paper3/v3n2-3pd.pdf

[2] Osborne, Jason A. *Markov Chains for the RISK Board Game Revisited Mathematics Magazine*, Vol. 76, No. 2, pp. 129-135, April 2003

[3] RISK: Strategy. Hasbro, Inc. On-line. http://www.hasbro.com/risk/default.cfm?page=strategy Accessed on April 15, 2008.

[4] RISK: The Game of Global Domination, Hasbro Inc., Pawtucket, RI, 1999.

[5] American Mathematical Society's Introductory Probability Book. Chapter 11. Online. Accessed on April 18, 2008.
http://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/Chapter11.pdhttp://www.dartmouth.edu/~chance/teaching_aids/books_articles/probability_book/Chapter11.pdf

Land of Oz:

$$\begin{array}{c} \\ R \\ N \\ S \end{array}\begin{array}{c} \begin{array}{ccc} R & N & S \end{array} \\ \begin{pmatrix} 1/2 & 1/4 & 1/4 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{pmatrix}\end{array} \qquad P^2 = \begin{array}{c} \\ R \\ N \\ S \end{array}\begin{array}{c} \begin{array}{ccc} R & N & S \end{array} \\ \begin{pmatrix} .438 & .188 & .375 \\ .375 & .25 & .375 \\ .375 & .188 & .438 \end{pmatrix}\end{array}$$

Drunkard's walk:

$$\begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}\end{array}$$

canonical form:

$$P = \begin{array}{c} \\ TR \\ ABS \end{array}\begin{array}{c} \begin{array}{cc} TR & ABS \end{array} \\ \left(\begin{array}{c|c} Q & R \\ \hline 0 & I_n \end{array}\right)\end{array} \qquad P^n = \begin{array}{c} \\ TR \\ ABS \end{array}\begin{array}{c} \begin{array}{cc} TR & ABS \end{array} \\ \left(\begin{array}{c|c} Q^n & * \\ \hline 0 & I_n \end{array}\right)\end{array}$$

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 0 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 0 & 4 \end{array} \\ \left(\begin{array}{ccc|cc} 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ \hline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right)\end{array}$$

The sequence $\{r^n\}$ is convergent if $-1 < r \leq 1$

$\lim_{n \to \infty} r^n = 0$ if $-1 < r < 1$

Every bounded, monotonic sequence is convergent.

If the series $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n \to \infty} a_n = 0$

toss a coin, and let $x_1 = 1$ if outcome is H and 2 if it is T
then toss x_1 coins and let x_2 = total # of H from beginning
+ 2 (total # of T from beginning)
then toss x_2 coins, and let x_3 = (total # of H from beginning)
+ 2 (total # of T from beginning)
As you see, X_{n+1} depends not only on X_n but also
on x_1, x_2, x_3