Rita Stevens
Claire Beebe

Painting the Plane

Present the Problem:

What is the minimum number of colors for painting the plane so that no two points at unit distance receive the same color?

Introduction to chromatic number

The chromatic number of a graph is the number of different colors which are needed to paint the vertices of the graph so that no two adjacent vertices are painted the same color.[1] For example, coloring two points one unit apart on a graph require them to be two different colors and painting a line segment requires at least two colors in order to paint the two vertices different colors. In order to create a graph which uses only two colors (a bicolorable graph) it is necessary to equivalently draw a bipartite graph. A bipartite is a set of vertices in which no two graph vertices within the same set are adjacent.

The chromatic number for many common, simple graphs can be easily evaluated. For example, a wheel graph W_n has chromatic number three for n odd and four for n even. A wheel graph of order n contains a cycle of order n-1 (essentially n-1 spokes) and every outer vertex is connected to an adjacent outer vertex.[2] A cycle graph C_n contains a cycle through all n nodes where the chromatic number is three for n odd and two for n even. An n-star graph is formed from n+1 nodes from the same vertex.[3]

History of the Problem:

There is varying information about who and when this problem was posed. Three men are mentioned as having posed this question. Edward Nelson and Hadwider are said to have come up with this problem in 1944.[4] Edward Nelson is said to have posed it in

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[1] PlanetMath.org – Chromatic Number

[2] Math World – Wheel Graph

[3] Math World – Star Graph

[4] *Problem 57: Chromatic Number of the Plane.*

1950.[5] And M. Gardner and Hadwiger are said to have posed this problem in 1960-61.[6] Basically, it was posed in the mid 1900s.

Erdös and de Bruijn posed related problems in which they "showed that the chromatic number of the plane is attained for some finite subgraph of G," which is "the infinite unit-distance graph G, with every point in the plane a vertex, and an edge between two vertices if they are separated by a unit distance."[4] This led to the narrowing of the answer to the problem to the chromatic number being between 4 and 7.[4] In his 1970 paper, Dmitry Raiskii proved "that the minimal number of colors in coloring of the plane is 4, 5, or 6, if we require each color to forbid a distance," who also happened to be a high school student.[5]

Explain the Problem:

Painting the plane is also known as the chromatic number of the plane and falls under the combinatorial geometry category. It also falls under the Ramsey theory branch of mathematics, named after Frank P. Ramsey, "the mathematical study of combinatorial objects in which a certain degree of order must occur as the scale of the object becomes large."[7] It "seeks regularity amid disorder."[8] The problems in Ramsey theory "typically ask a question of the form: how many elements of some structure must there be to guarantee that a particular property will hold?"[9]

"An example of a problem in Ramsey theory is the pigeonhole principle. Suppose that we know that n pigeons have been housed in m pigeonholes. How big must n be before we can be sure that at least one pigeonhole houses at least two pigeons? The answer is the pigeonhole principle: if $n > m$, then at least one pigeonhole will have at least two pigeons in it."[9]

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[5] *Stephen P. Townsend's 1979 Proof.*

[6] *Old and New Unsolved Problems in Plane Geometry and Number Theory*

[7] Math World – Ramsey Theory

[8] Wikipedia – Ramsey's Theorem

[9] Wikipedia – Ramsey Theory

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"A typical result in Ramsey theory starts with some mathematical structure, which is then cut into pieces. How big must the original structure be, in order to ensure that at least one of the pieces has a given interesting property?"[9]

This problem is similar (reminiscent) of the four-color map problem. In this problem "a 'map' is formed from a finite number of non-overlapping plane regions ('countries'), each of which is to be painted with a single color in such a way that different colors are used for any two countries that have a common boundary. It was conjectured in 1852 that four colors suffice to paint every planar map in this way," but this wasn't established until 1976.[6] If this problem takes that long to be settled, "we should know the answer by the year 2084" or about then since it was posed in the mid 1900s.[6]

Boundaries of the chromatic number of the plane

Since it is known that the minimum number of colors necessary to paint the plane in such a way as described by the problem is at least four, it is necessary to present a proof to explain why it is not possible to color the plane using only two or three colors. Obviously two colors will not suffice to paint the plane because of the above explanation of the chromatic number of a line. In order to that the plane cannot be painted using three colors, this we will assume that the plane can be colored in this way using red, green, and blue so that no two points one unit apart receive the same color and arrive at a contradiction, which was first proved by Edward Nelson.

Using only three colors it is obviously necessary to begin with an equilateral triangle with sides of length one to assure that each of the three colors is at least one unit away from any other point of the same color. Paint each of the vertices one of the three colors: one red, one green, one blue where the red vertex is labeled as point r, green as point g, and blue as point b. If we create a rhombus using two equilateral triangles (one as stated above) then it is necessary to paint the opposite vertices (ones which are not included in both triangles) the same color. So in our example if the green and blue vertices are shared by both triangles then it is necessary for the opposite vertices to both be red. Label the new red vertex r'. Rotating the rhombus around point r we can create essentially a new rhombus with points r, b', g', r'' in which r' from the original rhombus and r'' are within one unit of each other (this construction is called a Moser Spindle). Since the two red points (and also the two green and two blue points) are within one unit of each other then this ends our proof. The chromatic number of the Moser Spindle is actually four. It was first established by Dmitry Raiskii (a high school student) that the chromatic number of the plane was either four, five, or six.[6]

Similarly it is known that the maximum number of colors necessary to paint the plane is seven and so a proof of this is necessary to establish firmly the boundaries of the problem. This upper bound of seven was first proved by John Isbell. By tiling the plane with regular hexagons in which the maximum distance between any two points is slightly less than one unit and painting them carefully, it is possible to tile the plane in such a way which satisfies our problem. Painting the hexagons using seven different colors, choose one hexagon and paint it color 1 and then its surrounding hexagons paint colors 2-7. Using the resulting 18-sided figure as a model, paint the rest of the hexagons in this way so that the plane is essentially tiled using 7 hexagon clusters which are exactly the same.

Example Problems

Consider the tiling of the plane in which the basic "prototile" is a parallelogram that is the union of four regular hexagons and eight equilateral triangles, all of side-length 1. Let these twelve figures be painted in colors numbered 1-6 and extend by translation to the entire plane. Each triangle is to omit its boundaries (i.e. colors 5 and 6 are not used on the boundaries) and each hexagon omits its two lowest vertices and its rightmost vertex. Show that the distance 1 is not realized within any of the sets 1, 2, 3, 4 and the distance 2 is not realized within either of the sets 5 and 6.

Sets 1, 2, 3, 4 obviously do not realize a distance of one because since each of the side lengths of the given hexagons is one, this means that the diameter of the hexagon is greater than one. Therefore two points of the same color can be found within the same hexagon, but not between two hexagons. A distance of one between sets 5 and 6 can be realized since the boundaries of the triangles are omitted. A distance of two, however, between sets 5 and 6 cannot be realized since the distance between the two nearest triangles of the same color is only slightly greater than one.

References

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Math Capstone Coloring Page

Consider the tiling of the plane in which the basic "prototile" is a parallelogram that is the union of four regular hexagons and eight equilateral triangles, all of side length 1. Let these twelve figures be painted in colors 1-6 and then extend by translation to the entire plane. Use this to show that a distance 1 is not realized within sets 1-4 and a distance 2 is not realized within 5 or 6. (That two points of the same color are unit distance apart in sets 1-4 and 2 units apart in sets 5 and 6).
**Be careful at the boundaries; each triangle is to omit its edges and vertices, and each hexagon omits its two lowest vertices and its rightmost vertex.

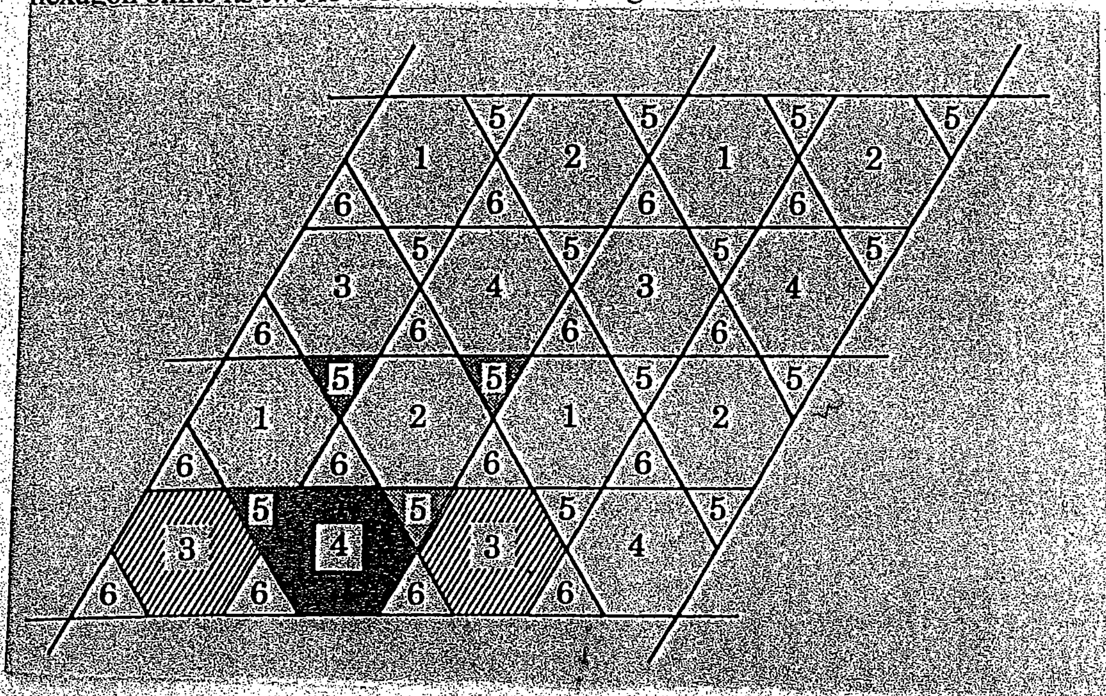

Fig 2

each side of hex or Δ is 1

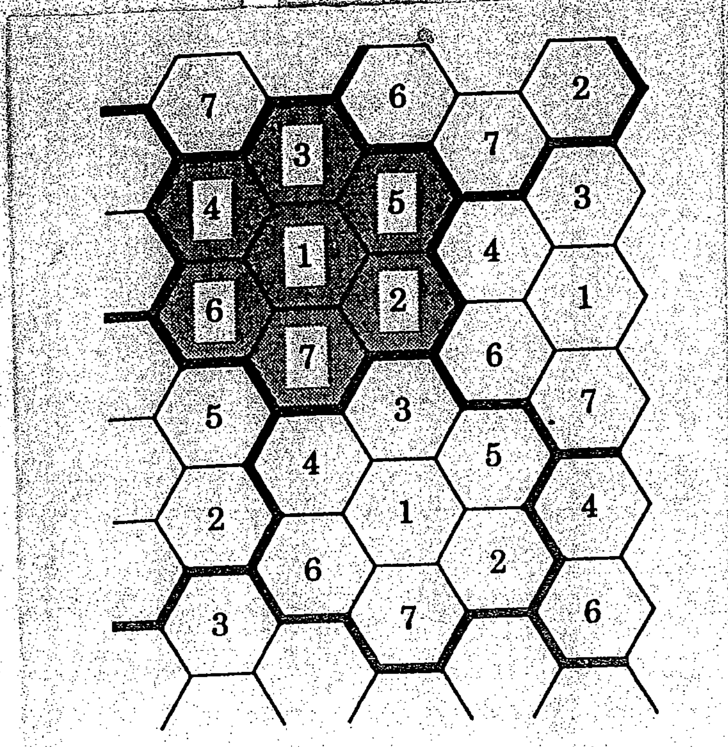

Fig 1

~~max distance~~ diam of each hexagon < 1

18-side figure T tiles plane.