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The Happy End Problem: A Tale of Romance

Introduction

The happy end problem was first posed by Ester Klein in 1935. Klein was a member of the Anonymous society, so called because it would meet under the statue of Anonymous in the city park of downtown Budapest, in Hungary.

Klein was doodling geometric figures on a piece of paper one day when she came up with the following question:

For a figure of n sides, how many points are required to guarantee that convex n-gon exists (assuming no three of the points are on the same line)?

This is usually represented as $g(n) = k$ where n is the number of sides and k is the number of points needed to guarantee the existence of a convex n-gon.

Obviously, the proof of g(3) is trivial: for any three points in a plane not on a line, it is possible to connect them and form a triangle. Therefore $g(3) = 3$, as shown in figure 1.

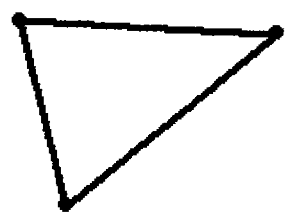

Fig. 1 - $g(3) = 3$

For $g(4) = 5$, Klein showed that all possible arrangements of five point could only fall into three different cases. In the first case, the five points form a convex polygon, and any four points will form a convex quadrilateral. In the second case, four of the points form the quadrilateral with one point in the middle. Finally, in the third case, the five points form a triangle with two points in the middle. If you draw a line through the two points, one side of the line will have two points on it. If you connect these two points with the two points on the line, a convex quadrilateral is formed, as shown in figure 2.

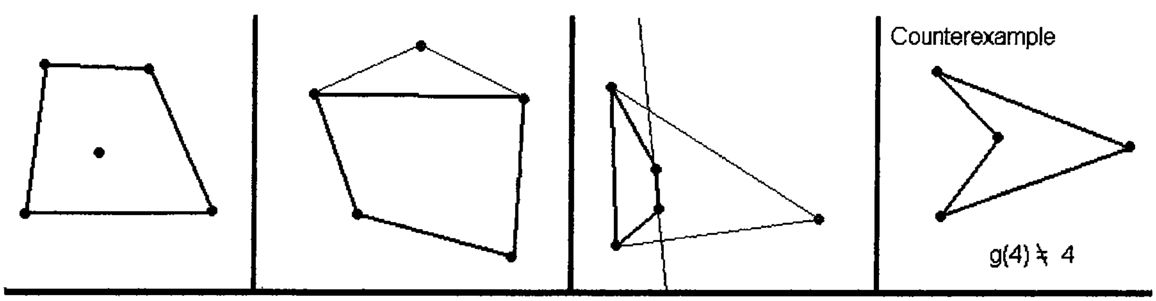

Fig. 2 - $g(4) = 5$

Three of the other members of the group, György Szekeres, Paul Erdös, and Endre Makai took an interest in this problem. Szekeres later admitted that his interest was motivated mostly by his other interest in Ester Klein.

Makai managed to prove that g(5) = 9. For almost any way of drawing eight points, a convex pentagon can be found. In fact, there is only one way to position eight points so that a convex pentagon cannot be formed.

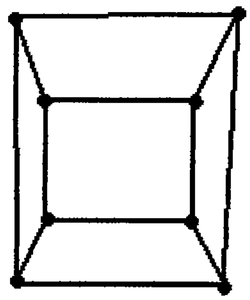

Fig. 3 - Counterexample for g(5) = 8

If point is added anywhere to this set of eight points, there will exist a convex pentagon. Therefore, g(5) = 9.

To date, no one has found an answer for g(6), but it is known to be bounded by 17 < g(6) < 37.

Although Szekeres and Erdös could not find any other value of g(n), they did come up with an upper and lower bound for g(n). They actually came up with two different proofs to find boundaries. The first of these involves Ramsey theory, while the second is a more geometric proof.

Ramsey theory is a topic in combinatorics, the mathematics of counting. Specifically, Ramsey theory represents a way of counting the number of elements that there must be in a set so that a particular property will hold. In our case, we need it to count the number of elements needed so that there exists a convex n-gon.

Szekeres and Erdös managed to use Ramsey theory to prove that n exists for all possible n-gons. In addition, they were able to show that an upper limit for the sized of g(n) is <= R3(n,n). [Where 2^(bn^2) <= R3(n,n) < 2^(c^n) for some constants b and c.]

Upper Boundary for g(n)

A better upper bound can be found using the more geometric method. If we assume an (x,y) coordinate system for the plane, we define X to be the set of all points on the plane, such that X = {(x1,y1),(x2,y2),..., (xm, ym)}.

A subset of points is called an *r-cup* if $x_1 < x_2 < x_3 < \ldots < x_r$ and

$$(y_{i1} - y_{i2}) / (x_{i1} - x_{i2}) < (y_{i1} - y_{i2}) / (x_{i1} - x_{i2}) < \ldots < (y_{i(r-1)} - y_{ir}) / (x_{i(r-1)} - x_{ir})$$

A sub set of points is called a r-cap if $x_1 < x_2 < x_3 < \ldots < x_r$ and
$(y_{i1} - y_{i2}) / (x_{i1} - x_{i2}) > (y_{i1} - y_{i2}) / (x_{i1} - x_{i2}) > \ldots > (y_{i(r-1)} - y_{ir}) / (x_{i(r-1)} - x_{ir})$

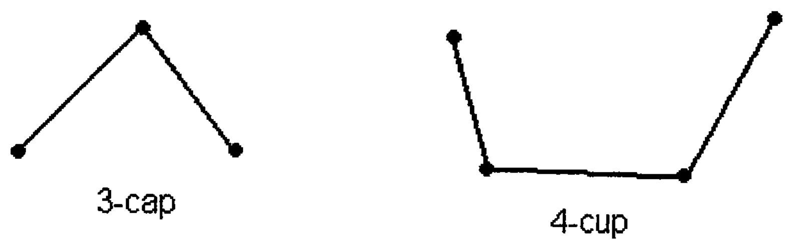

Fig. 4 - Caps and Cups

We define f(k,l) to be the smallest positive integer for which X contains a k-cup or an l-cap when X has at least f(k,l) points.

First, observe that f(t,3) = f(3,t) = t. That is, in order to guarantee that a t-cap/cup or a 3-cap/cup exits, k points are needed. This is because if you have fewer than t points, all the points could be forming a single cap or cup that was not the 3-cap/cup, so neither condition would be reached.

Secondly, we claim that f(k,l) <= f(k,l-1) + f(k-1,l) - 1.
<u>Proof:</u> Suppose X contains f(k,l-1) + f(k-1,l) - 1 points. Let Y be the set of left endpoints of (k-1)-cups in X. If X\Y contains f(k-1,l) points, then X/Y contains an l-cap, and X must contain an l-cap. Otherwise, Y must contain f(k,l-1) points. Suppose Y contains an (l-1)-cap {(xi1,yi1), (xi2, yi2), …, (xil-1, yil-1)}. Let {(xj1,yj1), (xj2, yj2), …, (xij-1, yij-1)} be a (k-1)-cup with il-1 = j1. A quick sketch then shows that either (xil-2,yil-2) can be added to the (k-1)-cup to create a k-cup or (xj2, yj2) can be added to the (l-1)-cap to create an l-cap.

We can show that f(k,l) <= (k+l-4) nCr (k-2) + 1 for all f(t,3) and f(3,t). We can also show that f(k,l) <= f(k,l-1) + f(k-1,l) - 1 holds for f(k,l) <= (k+l-4) nCr (k-2) + 1. Therefore, by induction, f(k,l) <= (k+l-4) nCr (k-2) + 1.

If an n-cup or n-cap exists, we can create a convex n-gon by connecting the first and last points of the cap or cup. So we know g(n) <= f(n,n) <= (2n-4) nCr (n-2) + 1.

Recently, in 1998, tighter upper bounds have been proved. Chung and Graham proved g(n) <= [(2n-4) nCr (n-2)] +7 - 2n. Shortly thereafter, Klietman and Patcher proved g(n) <= [(2n-5) nCr (n-2)] +2.

Proof of the Lower Boundary for g(n)

First, we claim that $f(k,l) = (k+l-4)\ nCr\ (k-2) + 1$

Proof:
We can easily show that $f(k,3) = f(k,3) >= (k+3-4)\ nCr\ (k-2)$.

We claim that the recurrence $f(k,l) > ((k-1) + l - 4)\ nCr\ ((k-1) - 2) + (k + (l - 1) - 4)\ nCr\ (k - 2)$.

We know this is true for the base cases. For induction, let us call the first nCr group A and the second B. We start with A not containing a (k-1)-cup or l-cap and B not containing a k-cup or (l-1)-cap.

Translate A and B until:
(i) every point of B has a first coordinate greater than that of any first coordinate in A
(ii) the slope of any line from a point in A to a point in B is greater than the slope of any line connecting points of A or points in B.

Let X = A U B. Any cup in C in A and B can only contain one element in B. Therefore X contains no k-cup. Similarly, X can contain no l-cap.

So the recurrence $f(k,l) > [((k-1) + l - 4)\ nCr\ ((k-1) - 2)] + [(k + (l - 1) - 4)\ nCr\ (k - 2)]$ must be true. Therefore $f(k,l) >= ((k-1) + l - 4)\ nCr\ ((k-1) - 2) + (k + (l - 1) - 4)\ nCr\ (k - 2) + 1$ which is equal to $f(k,l) >= (k+l-4)\ nCr\ (k-2) + 1$. Since we have already shown that $f(k,l) <= (k+l-4)\ nCr\ (k-2) + 1$, we can say that this relationship is actually equality. Q. E. D.

Second, we can use this equality to demonstrate that $g(n) >= 2^{(n-2)} + 1$.

Proof: We will construct a set of $2^{(n-2)}$ points with no subset of n points in convex position.

For $i= 0,1,\ldots,n-2$, let T_i be a set of $[(n-2)\ nCr\ (i)]$ points with no (i+2)-cap and no (n-i)-cup. No two points in the set can be connected by a line having a slope greater than 1. We know we can have this many points because of the equality we just proved.
($f(n-i,i+2) = [(n-2)\ nCr\ (n-i-2)] + 1 = [(n-2)\ nCr\ (i)] + 1 > [(n-2)\ nCr\ (i)]$)

For $i= 0, 1,\ldots,n-2$, place a small copy of T_i in a neighborhood of the point on the unit circle making an angle of $\pi/4 - i*\pi/(2*(n-2))$ with the positive x-axis.

Let X = the union of T_i for $i=0,1,\ldots n-2$, so $|X| = sum((n-2)\ nCr\ (i)) = 2^{(n-2)}$

Suppose Y is a subset of X in convex position. Let k and l be the smallest and the largest values of I so that Y intersection T_i is not empty.

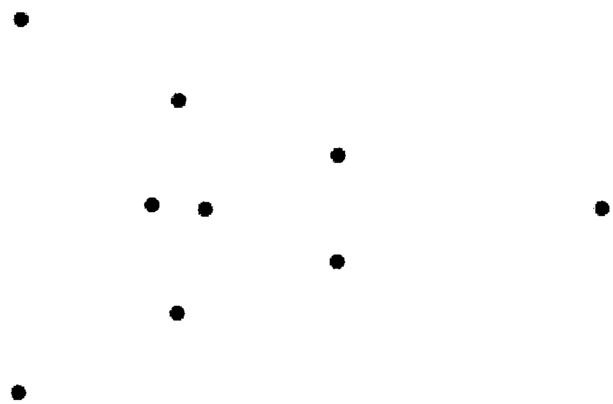

Fig. 6

Trivially, $H(3) = 3$, and from Figure 5 we easily conclude that $H(4) = 5$. Using a direct geometric approach, Harborth showed in 1978 that $H(5) = 10$. The inequality $H(5) \geq 10$ immediately follows from Figure 6, where a set of nine points in general position determines no empty convex pentagon (there are still two convex pentagons, neither being empty). Then Harborth uses Makai's Erdös-Szekeres theorem-related result to say that since $N(5) = 9$ then for every set of points X_n for $n \geq 10$ there must be at least one convex pentagon which is not necessarily empty. If the pentagon is non-empty, then Harborth considers several cases which systematically narrow the margin in which H(5) lies. First, if there are $m \geq 2$ points inside the pentagon, then we know that two of the m points and three of the points on the pentagon form another pentagon. If the smaller pentagon is non-empty, we find another two points inside that one, form another pentagon, and so on. Eventually we will find a convex pentagon P with either zero or one point inside it. In the case where there is one point inside the convex pentagon, that one point forms quadrilaterals with the vertices of the pentagons. Only if all these quadrilaterals are convex is there an empty convex pentagon within the convex pentagon. Otherwise, Harborth dissects the plane into regions and shows that if there are points in any of those regions then there must be an empty convex pentagon in X_n. In the end, $H(5) = 10$.

In 1983 Horton showed that H(n) does not exist for all $n \geq 7$. This statement is due to the following analytic construction of a planar set S_k of 2^k $(k \geq 1)$ points in general position determining no empty convex 7-gon. Let $a_1 a_2 \cdots a_k$ be the binary representation of the integer i, $0 \leq i < 2k$, where leading 0's are omitted. Put $c = 2^k+1$ and define $d(i) = \sum_{j=0}^{k} a_j c^{j-1}$. Now a simple analytical consideration shows that any convex polygon determined by the set $S_k = \{(i, d(i)) : i = 0, 1, \ldots, 2^k - 1\}$ has at most six vertices.

Valtr defines a Horton set inductively as follows. The empty set and any one-point set are Horton sets. The points of a Horton set H are in general position in the plane, with distinct x-coordinates. Furthermore, H can be partitioned into two sets A and B such that:

1. Each of A and B is a Horton set.

2. The set A is below any line connecting two points of B, and the set B lies above any line connecting two points of A.
3. The x-coordinates of the points of A and B alternate.

One can easily prove by induction on $n = |H|$ that if $\{(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4)\}$ is a 4-cup (respectively, 4-cap) in H, then there is a point (x, y) of H lying above (respectively, below) one of the segments $[(x_i, y_i); (x_{i+1}, y_{i+1})]$, $i = 1, 2, 3$. It immediately follows that H contains no empty 7-gon, because otherwise A would contain a 4-cup or B would contain a 4-cap.

Note that the above sets S_k fit the definition of Horton sets. Valtr uses Horton sets in several generalizations of the empty polygon problem.

What about H(6)? Does it exist?

Horton expresses the belief that H(6) exists. Bárány and Valtr present a conjecture which would imply the existence of H(6). Trying to determine the lower bound on H(6), Avis and Rappaport elaborated a method to determine whether a given set of points in the plane contains an empty convex 6-gon, and by using this approach they found a set of 20 points in general position containing no empty convex 6-gon.

Mark Overmars, a university professor in The Netherlands, constructed an algorithm in 1989 of time complexity $O(n^2)$ that solves the following problem: for a given set V in the plane, containing no empty convex 6-gon, and for a point $z \notin V$, determine whether the set $\{z\} \cup V$ contains an empty convex 6-gon. Using this algorithm, they found a set of 26 points containing no empty convex 6-gon. Hence, $H(6) \geq 27$, if it exists.

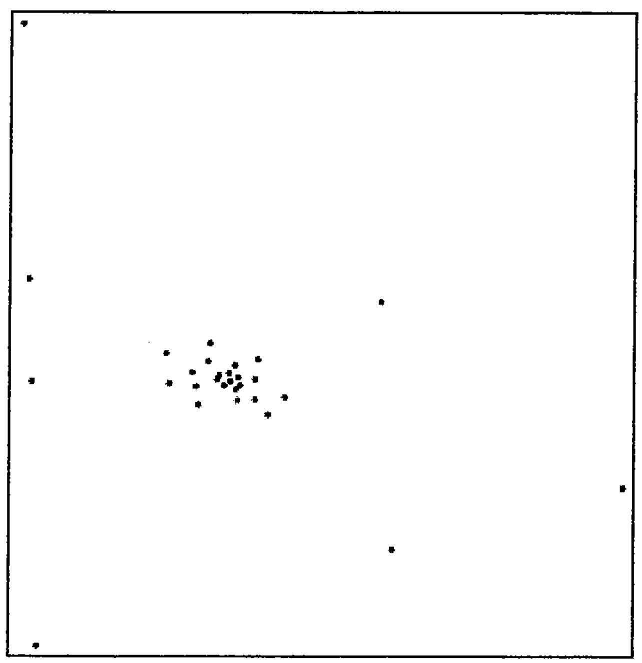

Fig. 7

With the use of more modern technology, Professor Overmars developed a computer program in 2003, compatible with Window 95 through XP, which can be downloaded at the following website: http://www.cs.uu.nl/people/markov. Using this program, the largest set of points found so far is 29 as displayed in Figure 7 above. So, today we know that $H(6) \geq 30$, if it exists.

Along with the original Erdös-Szekeres problem, the theory for the empty polygon problem remains incomplete and will require some new techniques to be solved.

Works Cited

Weisstein. "Happy End Problem." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/HappyEndProblem.html

Morris and Soltan. "The Erdos-Szekeres Problem on Points in Convex Position." From *AMS*. http://www.ams.org/bull/2000-37-04/S0273-0979-00-00877-6/S0273-0979-00-00877-6.pdf

Nelson. "On the Erdos Problem of Empty Convex Hexagons." http://www.math.hmc.edu/seniorthesis/archives/2004/jnelson/jnelson-2004-thesis.pdf