Notes — The Quest to Calculate Pi

10/21/2005

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Introduction
- What is π?
- What do we know about π?
- Why pi?, how do you calculate π?

Old School π
- Babylonian method
- Egyptian method
- Archimedes (250 BC – 17th c.)

New School π
- Arctan method (17th c. — 1980s)
- Computers (1980–)

Conclusion
- What don't we know about π?

Intro

Today John Paul and I will be talking about π. The Quest to Calculate π is a journey which has spanned continents & millenia. I'm sure our title has already riveted your attention to our every word and I'm sure a plethora of questions are racing through your head.

What is π?

have you asked yourself — $4\arctan 1 = \pi$ $\quad \dfrac{\pi^2}{6} = \dfrac{1}{1^2} + \dfrac{1}{2^2} + \dfrac{1}{3^2} + \dfrac{1}{4^2} + \cdots$

we use it all the time in math: $\pi r^2 = A$; $d\pi = C$; $e^{\pi i} + 1 = 0$

3.1415 926535 8979323846 2643383279 5028841971

it shows up in normalization of normal distribution, ~~an~~ the distribution of primes, all over the place

a quick answer: all geometric definitions equivalent to $d\pi = C$ — known first

first knowledge of π: $d\pi = C$, but many other places it shows up

$\therefore$ kind of a mystery which is one of the reasons π is so cool

might also ask yourself...

What exactly do we know about π?

π, ratio of ~~dia~~ circumference of a circle to its diameter is...

irrational (Johann Heinrich Lambert 1761)
transcendental (Ferdinand von Lindemann 1882) $\Big\}$ means...

can't construct, square the circle

we know

- 1st 0 in decimal expansion — pos. 32
- 1st time all 10 numbers occur in a 10 digit block pos. 60
- in order 0123456789 : pos. 17,387,594,880
- self referential positions : 1; 16,470; 44,899; 79,873,884
- Feynman points : six 9s position 768
- decimal digits past 1 trillion 242 million
  programs to find birthdays, messages, addresses, etc.

but still much we do not know
might also ask..

Why π? — why keep searching for ways to find more digits
10 or fewer digits enough significant figures for all practical calculations
39 - sufficient to calculate volume of universe (π unleashed)

it posseses people

Dutch mathematician Ludolph van Ceulen so proud of the 35 digits of π he calculated, he had it put on his tombstone

why?

- test computer systems, find bugs
- classical and ultimate testbed for numeric analysis
- unsolved questions raised by π in many fields
  makes it more interesting
- set world records

How do you calculate π?

today we'll try to give you a snapshot of how it has been done over time

My colleague John Paul Cook will now cover early attempts to calculate π — old school π if you will.

JPC

Babylonians
Egyptians
Archimedes

I write:

Babylonians $3\frac{1}{8} = 3.125$

Egyptians $4\left(\frac{8}{9}\right)^2 = 3.1604$

Indians $4\left(\frac{9785}{11136}\right)^2 = 3.088$

Bible 3

Plato $\sqrt{2}+\sqrt{3} = 3.14626$

Archimedes $3\frac{10}{71} < \pi < 3\frac{1}{7}$

$3.1408 < \pi < 3.14285$

New School π

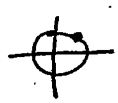
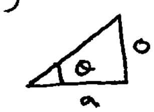
- Archimedean method used to calculate π, expand # of digits known for 2000 years
- then, in second half of 17^{th} century, infinitesimal analysis developed

  infinitesimal analysis — archaic term for calculus

  remember 3rd calculus class — series, go back

  in 17th century there emerged
- formulations of infinite expressions for π using series based on the arctan function

arctan x inverse function of $\tan x$ on $(-\pi/2, \pi/2)$

on unit circle  $\frac{\sin x}{\cos x} = \frac{y}{x}$ 

the point $\pi/4$ on unit circle
angle $\pi/4$ in triangle $\Big\}$ interesting for our purposes

$\tan \pi/4 = \frac{\pi/4}{\pi/4} = 1$; $o = a \Rightarrow \frac{o}{o} = 1$

so, going back to arctan formula arctan $1 = \frac{\pi}{4} \Rightarrow \pi = 4 \arctan 1$

we can calculate this thanks to the infinite series for arctan

discovered by James Gregory 1671

Gregory series $\arctan x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$

substituting 1, we get the Leibniz series 1674 $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} = 3.3396$

$$\arctan 1 = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2n+1}$$

this doesn't converge quickly : 2 billion terms — 9 accurate decimal places of π

quick convergence = finding digits of π with less work

so, attempts were made to assemble angle $\frac{\pi}{4}$ from relatively shorter arcs

Simplest: Euler 1738 $\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}$

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$$

substitute $\tan \alpha = \frac{1}{2}$, $\arctan \frac{1}{2} = \alpha$

$\tan \beta = \frac{1}{3}$; $\arctan \frac{1}{3} = \beta$

$$\frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\left(\frac{1}{3}\right)} = \frac{\frac{5}{6}}{1 - \frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1 \Rightarrow \tan\left(\tfrac{1}{2} + \tfrac{1}{3}\right) = 1 = \tan \frac{\pi}{4}$$

$$\frac{\pi}{4} = \arctan \frac{1}{2} + \arctan \frac{1}{3}$$

$$\arctan\frac{1}{2}+\arctan\frac{1}{3} = \frac{1}{2}+\frac{1}{3}-\frac{1}{2^3\cdot 3}-\frac{1}{3^3\cdot 3}+\frac{1}{2^5\cdot 5}+\frac{1}{3^5\cdot 5}-\frac{1}{2^7\cdot 7}-\frac{1}{3^7\cdot 7}+\cdots$$

$\frac{1}{2^9\cdot 9}+\frac{1}{3^9\cdot 9}$

3.14174

Machin formula

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha} \qquad \tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

use $\alpha = \arctan\frac{1}{5} = \frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{5^7\cdot 7}$ easy to calculate

$$\tan 2\alpha = \frac{5}{12} \qquad \tan 4\alpha = \frac{120}{119} \Rightarrow$$

$$\alpha = 4\arctan\frac{1}{5}$$

4α only slightly larger than $\frac{\pi}{4}$; can find angle β s.t. $\frac{\pi}{4} = 4\alpha - \beta$

$$\frac{\tan 4\alpha - \tan\frac{\pi}{4}}{1+\tan 4\alpha\cdot\tan\frac{\pi}{4}} = \frac{\frac{120}{119}-1}{1+\frac{120}{119}(1)} = \frac{\frac{1}{119}}{\frac{239}{119}} = \frac{1}{239}$$

$$\frac{\pi}{4} = 4\alpha - \beta = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

$$= 4\left[\frac{1}{5}-\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}-\cdots\right] - \left[\frac{1}{239}-\frac{1}{3\cdot 239^3}+\cdots\right]$$

wanted to give idea of convergence but calculation difficult

Others: Størmer 1896 $\frac{\pi}{4} = 6\arctan\frac{1}{8} + 2\arctan\frac{1}{57} + \arctan\frac{1}{239}$

Gauss used in 1973 $\frac{\pi}{4} = 12\arctan\frac{1}{18} + 8\arctan\frac{1}{57} - 5\arctan\frac{1}{239}$

these formulas used until 20-25 years ago

1940s advent of computing made ~~more~~ longer calculations possible

1980s FFT

computing power increasing all the while

other formulas – specific to computing

Ramanujan $$\frac{1}{\pi} = \frac{\sqrt{8}}{9801}\sum_{n=0}^{\infty}\frac{(4n)!}{(n!)^4}\cdot\frac{1103+26390n}{396^{4n}}$$

converges 4x faster

Chudnovsky 1987

see other paper

Gauss AGM

number of π digits doubles each iteration stage (5)

∴ converges quadratically

Borwein Quadratic

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all these decimal

hexadecimal expansion is

3.14159265

3.243F6 ~~A8885~~

BBP

do two or three digits

Conclusion JP Cook