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# The Perfect Cuboid Problem

Outline

- Introduction to the problem (James)
- Properties of the problem (Tuan)
- Jon Perry's (bogus) proof (James)
- What's wrong with the proof? (Tuan)
- Tuan's method for simplifying a computer search (Tuan)

**Introduction: Perfect Cuboid and Related Problems**

Pythagorean Triple Problem
In class we looked at the problem of finding integer solutions to the equation

$$x^2 + y^2 = z^2 .$$

From the Pythagorean Theorem we know that this problem can be expressed geometrically. The problem is equivalent to finding right triangles with integer sides.

{Draw Triangle}

We call any set of integers that satisfies these constraints a Pythagorean triple.

In class we derived the general Babylonian formula for all Pythagorean triples.

$$x = p^2 - q^2 \qquad y = 2pq \qquad z = p^2 + q^2 \quad \text{where } p, q \in Z$$

Euler Brick Problem
A natural extension of the Pythagorean triple problem into three dimensions is the Euler brick problem.

In the Euler brick problem we look for cuboids with integer sides and integer face diagonals.

Definition: A cuboid is a rectangular box. A solid three dimensional figure composed of six rectangular faces joined at right angles. The opposite faces of a cuboid are equal.

{Draw Cuboid}

The problem is equivalent to finding integer solutions to the following system of equations.

$$\begin{aligned} a^2 + b^2 &= d^2 \\ a^2 + c^2 &= e^2 . \\ b^2 + c^2 &= f^2 \end{aligned}$$

According to mathworld.com, interest in the Euler brick problem ran high in the 18th century. Many Euler bricks have been found. The smallest Euler brick was discovered by Paul Halcke in 1719. {a, b, c, d, e, f} = {240, 117, 44, 267, 244, 125}.

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Formulas have been derived that give a set of integers {a, b, c} which satisfy the system of equations, but no general formula for all {a, b, c} has been found. For example, consider the following formula discovered by Saunderson

$$a = x(4y^2 - z^2) \qquad b = y(4x^2 - z^2) \qquad c = 4xyz$$

where {x,y,z} are a Pythagorean triple (given by the equations for x, y, and z earlier in the paper).

Perfect Cuboid Problem

This brings us to the perfect cuboid problem. The problem is simply an extension of the Euler brick problem. Do there exist cuboids with integer sides, integer face diagonals, and an integer space diagonal?

{Draw the space diagonal on the Euler brick figure}

The problem is equivalent to finding integer solutions to the following system of equations

$$\begin{aligned} a^2 + b^2 &= d^2 \\ a^2 + c^2 &= e^2 \\ b^2 + c^2 &= f^2 \\ a^2 + b^2 + c^2 &= g^2 \end{aligned}$$

Does there exist a set of integers {a, b, c, d, e, f, g} that is a solution to this system of equations?

**Why a,b,& c have to be even, even, and odd**

1. $a^2+b^2=d^2$
2. $a^2+c^2=e^2$
3. $b^2+c^2=f^2$
4. $a^2+b^2+c^2=g^2$

Because $a^2+b^2+c^2=g^2$ can be written as three different Pythagorean triples, we can choose $d^2+c^2=g^2$ to generalize without loss of generality.

The Babylonian Pythagorean formula that we proved in class dictated that in any given Pythagorean triples, at least one of the legs has to be even (y=2pq). Thus, in the equation $d^2+c^2=g^2$, either d or c has to be even. Without loss of generality once again, we can choose d to be even. If d was not even and hence odd, then either a is odd or b is odd (meaning a and b cannot have the same parity). Then depending on if a or b was odd, either e or f has to be even. In either case, what we will generalize here will still hold.

So, without loss of generality: d is even. Now we consider equation 1.
Dictated by the Babylonian equation again, we see that either a or b has to be even. It does not matter which one is even because since d is even, the other side also has to be even. In other words, if a and d

are even, $a^2$ and $d^2$ are even, and their difference has to be even. Therefore, $b^2$ has to be even, and hence, b itself has to be even. Similarly, if e was even, then a and c have to be even; if f was even, then b and c have to be even.

So, given that a Pythagorean box exists, because of equation 4, we know that two out of the three sides have to be even, no matter what else is true about the box. In our specific example, a and b are the two sides and are always even. Now, we consider the last side, c. If c was even, then e, f, and g are all even, and thus, we can divide everything by 2 and still have a set of integers that obey equation 1 through 4. If c was still even after we had factored out a factor of 2, we still can divide everything again by 2 and have a Pythagorean box because a and b are still even. Hence, by the infinite descent rule, c cannot be infinitely even. In other words, if we consider a primitive Pythagorean box, one where a, b, c, d, e, f, & g do not have a common factor, then c has to be odd because a and b are still even. Similarly, if e was even, then a and c are even, and b is odd; if f was even, then b and c are even, and a is odd.

In summary, in a primitive Pythagorean box if it does indeed exist, two of the three sides are even, and the last side is odd.

Because one side is odd, it is twice as fast to run exhaustive search programs to find a set of integers that obey equations 1 through 4 (because they don't have to run the test for even numbers). In an exhaustive search, the program starts with an odd integer as a side and tests all combinations of even numbers less than the starting odd integer as the other two sides. For example, if c was 7, then the program would test with (2,2), (4,2), (4,4), and (6,4) to see if any of these (with c=5) would form a Pythagorean box. There is not a need to test for (2,4) and (4,6) because that's just switching a and b. According to MathWorld, they have test c up to $10^{10}$ or with c bigger than 10 billions and as of yet, no Pythagorean box has been found. This is indicative to us that it is extremely unlikely that our perfect cuboid or the Pythagorean box actually exists.

**Jon Perry's (bogus) Proof**

If a perfect cuboid exists, then Perry shows that its even face diagonal must be infinitely even (i.e. has infinite powers of 2). Since this is absurd, no perfect cuboid can exist.

If a perfect cuboid exists with sides a, b, and c, and the sides a, b, and c do not have factors in common, then two of these sides are even and the third is odd.

So, we can write the three sides of the cuboid as follows

$$a = 2s \qquad b = 2t \qquad c = 2u + 1.$$

If we expand the equation for the space diagonal g using these equations for a, b, and c, we get the following equation

$$\begin{aligned}
&a^2 + b^2 + c^2 = g^2 \\
&(2s)^2 + (2t)^2 + (2u+1)^2 = g^2 \\
&4s^2 + 4t^2 + 4u^2 + 4u + 1 = g^2 \\
&4(s^2 + t^2 + u^2 + u) + 1 = g^2
\end{aligned}$$

Next, note that the square of an even number is even and the square of an odd number is odd.

$$(2k)^2 = 4k = 2(k') \qquad (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2k'+1$$

Since sides a and b are even and side c is odd, the sum of the squares of these sides (i.e. the space diagonal squared: $g^2$) must be odd. Therefore, the space diagonal g must be odd.

An odd number squared has the form:

$$(2k+1)^2 = 4(k^2 + k) + 1$$

Since k is a natural number, the quantity $k^2 + k$ must be even. In the case of $g^2$ we have

$$4(s^2 + t^2 + u^2 + u) + 1 = 4(k^2 + k) + 1$$

So, the quantity $s^2 + t^2 + u^2 + u$ must be even. By the same reasoning that $k^2 + k$ is even, we know that $u^2 + u$ must be even. This means that for $s^2 + t^2 + u^2 + u$ to be even, the quantity $s^2 + t^2$ must be even.

For the quantity $s^2 + t^2$ to be even, s and t must both be odd or even (i.e. they must have the same parity).

Now, we note that

$$a^2 + b^2 = d^2$$
$$(2s)^2 + (2t)^2 = d^2$$
$$s^2 + t^2 = \frac{d^2}{4}$$

Because $s^2 + t^2$ is even, we know that $d^2 / 4$ is even.

**Here's where it goes wrong.**

So, $s = 2s'$ and $t = 2t'$. Thus, $a = 2s = 4s'$ and $b = 2t = 4t'$.

If we repeat the process with the new parameterization, then we arrive at:

$4[4s'^2 + 4t'^2 + u^2 + u] + 1 = g^2$

But now $d^2 / 16$ must be even, i.e. a = 8s'', b = 8b''.

Hence, by the infinite descent rule, this is obviously absurd. QED.

**Pythagorean Triples**

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# Pythagorean formula: $A^2+B^2=C^2$

**I. Defining the primordial constant A' and its characteristics**

Define $A'=C-B$

$A+B>C$ (the length of two sides of a triangle has to be greater than the other side)

$A>C-B=A'$

**II. Generating formula for B and C using A'**

$A^2+B^2=C^2$

$A^2+(C-A')^2=C^2$

$A^2+C^2-2CA'+A'^2=C^2$

$C=(A^2+A'^2)/(2A')$

Similarly, $B=(A^2-A'^2)/(2A')$

**III. Parity of A and A'**

$A^2=C^2-B^2$

If A is even, then $(\text{even})^2=C^2-B^2$.

If C is even, then B is also even.

If C is odd, then B is also odd.

Therefore, C-B=A' is always even if A is even.

Similarly, if A is odd, then A' is odd.

Thus, the parity of A and A' is always the same.

**IV. Case 1 where A is odd**

If A is odd, then we can generalize A in the form of $A=D^d*E^e*\ldots*P^p$, where D, E, …, P are prime factorization of A.

Since A' is also odd, let us generalize A' in the form of $A'=D^{d1}*E^{e1}*\ldots*P^{p1}$. We will deal with the case where A' has odd prime factors that A does not have at a later time.

Using the formula for C derived in part II, we have:

$$\begin{aligned} C &= (A^2+A'^2)/(2A') \\ &= (A^2/A'+A')/2 \\ &= (D^{2d-d1}*E^{2e-e1}*\ldots*P^{2p-p1}+D^{d1}*E^{e1}*\ldots*P^{p1})/2 \end{aligned}$$

Since all the prime factors of A, namely D,E,…,P, are all odd integers, as long as they are raised to a power greater than or equal to zero, then $D^{2d-d1}*E^{2e-e1}*\ldots*P^{2p-p1}+D^{d1}*E^{e1}*\ldots*P^{p1}$ will be even, which will be divisible by 2, and hence C will be an integer.

Thus, $2d-d1\geq 0$, $2e-e1\geq 0,\ldots,2p-p1\geq 0$.

Thus, $d1\leq 2d$, $e1\leq 2e,\ldots,p1\leq 2p$.

In other words, C will be an integer (which implies B will also be an integer, and hence we have a P triples) for any A' as long as the powers of each of the prime factors of A' do not exceed the powers of their respective prime factors in A by a factor of 2 and A'<A.

**V. Case 2 where A is even**

If A is even, then we can generalize A in the form of $A=2^n*D^d*E^e*\ldots*p^p$, where D, E, …, P are prime factorization of A.

Since A' is also even, let us generalize A' in the form of $A'=2^m*D^{d1}*E^{e1}*\ldots*P^{p1}$. We will deal with the case where A' has an odd prime factor that A does not have at a later time.

Using the formula for C derived in part II, we have:

$$
\begin{aligned}
C &= (A^2+A'^2)/(2A') \\
&= A^2/(2A')+A'/2 \\
&= (2^{2n}*D^{2d}*E^{2e}*\ldots*P^{2p})^2/(2*2^m*D^{d1}*E^{e1}*\ldots*P^{p1}) + 2^m*D^{d1}*Ee^{e1}*\ldots*P^{p1}/2 \\
&= (2^{2n-m-1}*D^{2d-d1}*E^{2e-e1}*\ldots*P^{2p-p1}) + 2^{m-1}*D^{d1}*E^{e1}*\ldots*P^{p1}
\end{aligned}
$$

If C is to be an integer, we then have limitations on the powers of each of the prime factors of A. From the first term, we see that $2n-m-1\geq 0$ and from the second term, we see that $m-1\geq 0$. Thus, $1\leq m\leq 2n-1$.
From the first term, we see that $2d-d1\geq 0$ and from the second term, we see that $d1\geq 0$. Thus, $0\leq d1\leq 2d$. Similarly, $0\leq e1\leq 2e, \ldots, 0\leq p1\leq 2p$.
In other words, C will be an integer (which implies that we have a P triples) for any A' as long as the powers of each of the odd prime factors of A' do not exceed the powers of their respective odd prime factors in A by a factor of 2 and the power of m is less than or equal to 2n-1 and A'<A.

**VI. Elimination of all other possibilities of A'**

Let's first take the case where A is odd and has the form of $A=D^d*E^e*\ldots*P^p$, where D, E, …, P are prime factorization of A and where A' has the form $A'=D^{d1}*E^{e1}*\ldots*P^{p1}*Q^{q1}*R^{r1}*\ldots*Z^{z1}$, where $D\neq E\neq\ldots\neq P\neq Q\neq R\neq\ldots\neq Z$. In other words, A' has prime factors that A does not have.
Using the formula for C derived in part II, we have:

$$C = ½*(A^2/A'+A')$$

The only term we care about is the $A^2/A'$ term. Since A' has odd prime factors that A does not have, $A^2/A'$ cannot be an integer. Thus, C cannot be an integer, and so we do not have a Pythagorean triple.

Similary, for the case where A is even and has the form of $A=2^n*D^d*E^e*\ldots*P^p$, where D, E, …, P are prime factorization of A and where A' has the form of $A'=2^m*D^{d1}*E^{e1}*\ldots*P^{p1}*Q^{q1}*R^{r1}*\ldots*Z^{z1}$, where $D\neq E\neq\ldots\neq P\neq Q\neq R\neq\ldots\neq Z$, C cannot be an integer.

Therefore, for any given integer A, in order to have a Pythagorean triples, we conclude that A' has to belong to either case IV or V and there is no other possibility for A'.
In other words, for any given integer A, all the possible Pythagorean triples are dictated by the generating formula derived in part II and are generated by the primordial constant A', which follows the rules derived from case IV and V.

Sources:

http://en.wikipedia.org/wiki/Euler_brick

http://mathworld.wolfram.com/EulerBrick.html

http://mathworld.wolfram.com/PerfectCuboid.html

http://www.users.globalnet.co.uk/~perry/maths/perfectcuboid/perfectcuboid.htm