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Math 4513

Presentation Summary

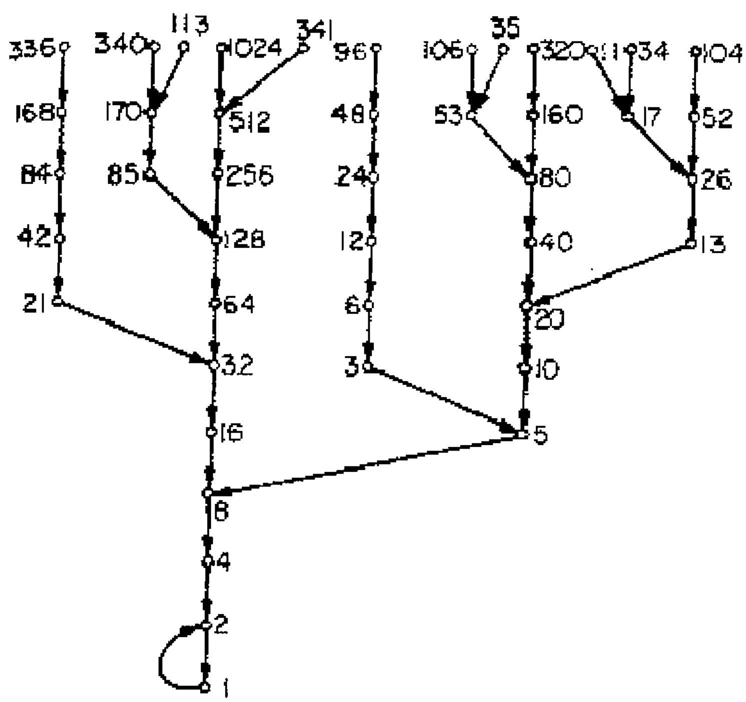
The "3n+1" problem is also known as Collatz's problem, the Syracuse problem, Kakutani's problem is a function such that if an integer n is odd then $3n+1$ and if the integer is even then $n/2$, then continue the steps iteratively until the function reaches 1. In the problems of unsolved number theory the question was asked do the iterations of all positive integers eventually reduce to 1, or rather does there exist a number that does not eventually reach 1? So far, it has not been proven so. In our presentation, we attempted to expand on some of the behaviors of numbers and to draw possible conclusions of the iterations.

To start we will define a slightly modified "3n+1" function as follows:

$$T^{(k)}(n) = \begin{cases} \dfrac{3n+1}{2}, & if \quad n \equiv 1 \pmod 2 \\ \\ \dfrac{n}{2}, & if \quad n \equiv 0 \pmod 2 \end{cases} \qquad \textbf{(2.1)}$$

Which basically says, if you have some number n, if it is odd(meaning it has a remainder of 1), you would input it into the first equation, if it is even(has no remainder) you input into the second equation until you arrive to the number "1."

Next, we will show an example of the iterations as follows:



This is also known as a Collatz graph.

Since this problem is unproven we will look at some of the conjectures other's have formulated in order to understand the behavior of the iterations for the "3n+1" or Collatz's problem. A conjecture is something that we think is true, not been proven yet. The first conjecture is as follows:

The Collatz graph of $T(n)$ *on the positive integers is weakly connected*

Which basically states that from the graph shown above the integers are weakly connected because weakly connected is defined as if it is connected when viewed as an undirected graph, i.e., for any two vertices there is a path of edges joining them, ignoring the directions on the edges (Lagarias). Moreover, we can call the sequence of iterates

$(n, T(n), T^{(2)}(n), T^{(3)}(n), \ldots)$ the trajectory of "n." With the trajectories there are 3 possible behaviors. They are as follows:

(i). *Convergent trajectory.* Some $T^{(k)}(n) = 1$.

(ii). *Non-trivial cyclic trajectory.* The sequence $T^{(k)}(n)$ eventually becomes periodic and $T^{(k)}(n) \neq 1$ for any $k \geq 1$.

(iii). *Divergent trajectory.* $\lim_{k\to\infty} T^{(k)}(n) = \infty$

The convergent trajectory which we already know, states that after a certain number of iterations, one of those iterations (mainly the final one) will produce a result of 1. Non-trivial cyclic trajectory, states that the iterations eventually becomes periodic, meaning it will never reduce down to 1 and goes up and down within the values produced. Divergent trajectory states, that through the iterations, it will never reduce to one and the values will continue to get larger into infinity. After stating the behaviors of the trajectories we can note that all the trajectories of positive "n" are convergent.

Next we will explain stopping time and total stopping time. Stopping time in iteration form can be defined as follows:

$$T^{(k)}(n) < n$$

or

$$\sigma(n)$$

Meaning, the iteration that produces a number less than n(the starting value) is said to be the stopping time. The total stopping time in iteration for is as follows:

$$T^{(k)}(n) = 1$$

or

$$\sigma_\infty(n)$$

Meaning the iteration that that produces a result of 1 is the total stopping time. After stating these two definitions we can now restate the conjecture in terms of stopping time. It is as follows: Every integer of "n" greater than two has a finite stopping time.

One of the striking features of this problem is the irregular behavior of the successive iterates. In a sense, it can be considered somewhat chaotic. We can measure the behavior of the iterates with use of stopping time and total stopping time with some called the expansion factor s(n).

$$s(n) = \frac{\sup_{k \geq 0} T^{(k)}(n)}{n}.$$

The expansion factor is an equation that takes the supreme value(largest value of the iterations) divided by the number n(the original starting value), if in is bounded. If it is not bounded then s(n) is equal to infinity and is divergent. One example of a bounded trajectory is the number 27. It produces an expansion factor:

$$s(27) = \frac{\sup_{k \geq 0} T^{(k)}(27)}{27} = \frac{4616}{27} \approx 171 .$$

Also, here is a break down of the iterates *n* =27 when put into an excel program.

k	$T^{(k)}(n)$
0	27
1	41
2	62
3	31
4	47
5	71
6	107
7	161
8	242
9	121
10	182
11	91
12	137
13	206
14	103
15	155
16	233
17	350
18	175
19	263
20	395
21	593
22	890
23	445
24	668

k	$T^{(k)}(n)$
25	334
26	167
27	251
28	377
29	566
30	283
31	425
32	638
33	319
34	479
35	719
36	1079
37	1619
38	2429
39	3644
40	1822
41	911
42	1367
43	2051
44	3077
45	4616
46	2308
47	1154
48	577
49	866

k	$T^{(k)}(n)$
50	433
51	650
52	325
53	488
54	244
55	122
56	61
57	92
58	46
59	23
60	35
61	53
62	80
63	40
64	20
65	10
66	5
67	8
68	4
69	2
70	1

The stopping time for n=27 was 59 and the total stopping time was 70. We can also note that the that the largest value was in the 45th iteration that produce a result of 4616. Nonetheless, here is a table showing the expansion factors of more numbers.

n	$\sigma(n)$	$\sigma_\infty(n)$	$s(n)$
1	∞	2	2
7	7	11	3.7
27	59	70	171.
$2^{50}-1$	143	383	6.37×10^8
2^{50}	1	50	1
$2^{50}+1$	2	223	1.50
$2^{500}-1$	1828	4331	1.11×10^{88}
$2^{500}+1$	2	2204	1.50

The irregular behavior of the trajectories can be seen between two number in this table. Just a difference of two numbers(2^500 -1 and 2^500 +1) has a big difference when it comes to computing the expansion factor. In reference to being chaotic, we can see that a small change in n produce large or small change in the expansion factor. Next Chase will explain the behavior of finite cycles.

The Finite Cycles Conjecture. *There are only a finite number of distinct cycles for the function T(n) iterated on the domain.* **Z**

First let us define the notation. For some n ϵ **Z** let we can define T(n) so that it returns either a 1 or 0 for $x_k(n)$ depending on whether it is odd or even repesctively for the kth iterate.

$$T^{(k)}(n) \equiv x_k(n) \pmod 2, 0 \le k < \infty \qquad \textbf{(2.2)}$$

For k=0 the function returns the following.

$T^{(0)}(n) = n$

The $x_k(n)$ can be expressed in the form of vector notation

$$v_k(n) = (x_0, \ldots, x_{k-1}(n)) \qquad \textbf{(2.3)}$$

The following function gives the result for the kth iteration for some given n

$$T^{(k)}(n) = \lambda_k(n)n + \rho_k(n) \text{ or in vector notation } T^{(k)}(n) = \lambda_k(v)n + \rho_k(v) \qquad \textbf{(2.4)}$$

Where

$$\lambda_k(n) = \frac{3^{x_0(n)+...+x_{k-1}(n)}}{2^k} \quad \textbf{(2.5)}$$

$$\rho_k(n) = \sum_{i=0}^{k-1} x_i(n) \frac{3^{x_{i+1}(n)+...+x_{k-1}(n)}}{2^{k-i}} \quad \textbf{(2.6)}$$

It can be shown that for any given length *k*, where *k* is the k^{th} iteration of the function *T(n)*, that there are only a finite number of possible solutions, in fact there only $\mathbf{2^k}$ **solutions** at most. This was shown by Böhm and Sontacchi. Böhm and Sontacchi show this by substituting this equation

$$T^{(k)}(n) = \lambda_k(n)n + \rho_k(n) \quad \textbf{(2.4)}$$

Into,

$$T^{(k)}(n) = n \ , \ n \in Z^+ \quad \textbf{(2.23)}$$

In order to obtain the following equation

$$(1 - \lambda_k(n))n = \rho_k(n)$$

$$\left(1 - \frac{3^{x_0(n)+...+x_{k-1}(n)}}{2^k}\right)n = \frac{3^{x_{i+1}(n)+...+x_{k-1}(n)}}{2^k} \sum_{i=0}^{k-1} x_i(n) \frac{2^i}{3^{x_0+...+x_i}} \quad \textbf{(2.24)}$$

$$n = \frac{3^{x_{i+1}(n)+...+x_{k-1}(n)}}{2^k} \left(1 - \frac{3^{x_0(n)+...+x_{k-1}(n)}}{2^k}\right)^{-1} \sum_{i=0}^{k-1} x_i(n) \frac{2^i}{3^{x_0+...+x_i}}$$

The result leaves n to be determined by a finite number of solutions since k is fixed. This condition holds and has a unique solution so long as the values for x_k are initially fixed and equation (2.23) is true.

Here is an example with some arbitrary initial odd number and with **k=3** and the vector sequence $v_3(\textit{starting w/ an odd number}) = (\textit{odd}, \textit{odd}, \textit{even}, \textit{odd})$ or by using equation **(2.3)** from above this can be expressed as $v_3(\textit{starting w/ an odd number}) = (1,1,0,1)$

$$T^{(3)}(starting\ w/\ an\ odd\ number) = \frac{\left(\frac{3\left(\frac{3n+1}{2}\right)+1}{2}\right)}{2}$$

$$= \frac{3^2 n}{2^3} + \frac{3}{2^3} + \frac{1}{2^2}$$

$$= \left(\frac{3^{1+1+0}}{2^3}\right)n + \frac{3}{2^3} + \frac{1}{2^2}$$

$$= \left(\frac{3^{1+1+0}}{2^3}\right)n + 1\frac{3^{1+0}}{2^3} + 1\frac{3^0}{2^2} + 0\frac{3^0}{2^1}$$

Thus by manipulating the above equation we now find it for a k length

$$T^{(k)}(n) = \frac{3^{x_0(n)+\ldots+x_{k-1}(n)}}{2^k}n + \sum_{i=0}^{k-1} x_i(n)\frac{3^{x_{i+1}(n)+\ldots+x_{k-1}(n)}}{2^{k-i}}$$

$$T^{(k)}(n) = \lambda_k(v)n + \rho_k(v)$$

This is the example I did in class **k=4** and **n=27** in order to familiarize one's self with the notation.

$T^{(0)}(27) = 27 \equiv 1$, $so\ x_0 = 1$

$T^{(1)}(27) = 41 \equiv 1$, $so\ x_1 = 1$

$T^{(2)}(27) = 62 \equiv 0$, $so\ x_2 = 1$

$T^{(3)}(27) = 31 \equiv 1$, $so\ x_3 = 1$

$T^{(4)}(27) = 47 \equiv 1$, $so\ x_4 = 1$

$$\lambda_4(27) = \frac{3^{1+1+0+1}}{2^4} = \frac{27}{16}$$

$$\rho_k(n) = 1\frac{3^{1+0+1}}{2^4} + 1\frac{3^{0+1}}{2^3} + 0\frac{3^1}{2^2} + 1\frac{3^0}{2^1} = \frac{23}{16}$$

$$T^{(4)}(27) = \frac{27}{16}27 + \frac{29}{16} = 47$$

Sources:

http://www.cecm.sfu.ca/organics/papers/lagarias/

http://mathworld.wolfram.com/CollatzProblem.html

http://www.numbertheory.org/pdfs/survey.pdf