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I. Power Distance

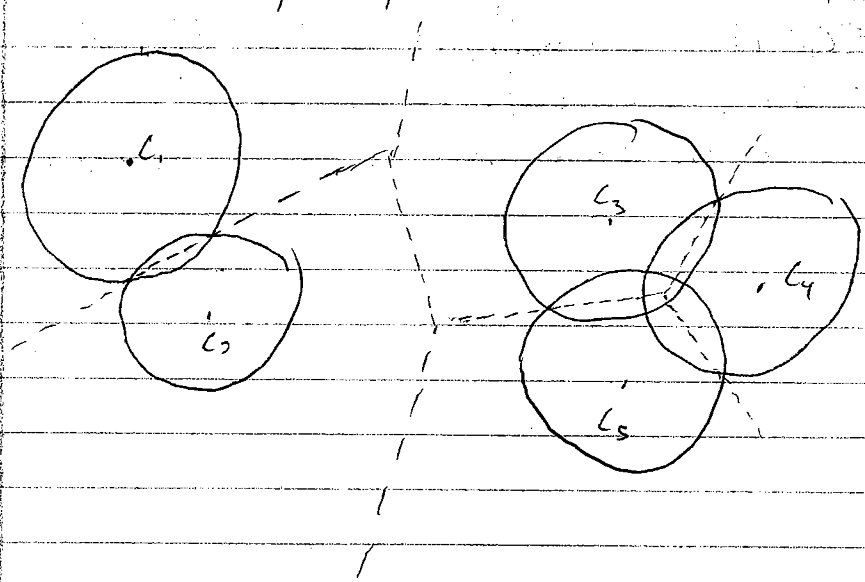
Given a disk on the plane ($\mathbb{R}^2$) with center c and any point $p \in \mathbb{R}^2$, we say the power distance from p to c = $|pc|^2 + r^2$ where r is the radius of the disk and $|xy| = dist(x,y)$

II. Power Diagram

Given numerous disks $D_1, \dots, D_n$ with centers $c_1, \dots, c_n$ and corresponding radii $r_1, \dots, r_n$, we can divide the plane into cells via the following description. If $p \in \mathbb{R}^2$ and $|pc_i|^2 - r_i^2 < |pc_j|^2 - r_j^2$ and $i \neq j$, then $p \in c_i$'s cell.

When two disks have a non-empty intersection, the power diagram cells are divided along the mutual chord of the circles.

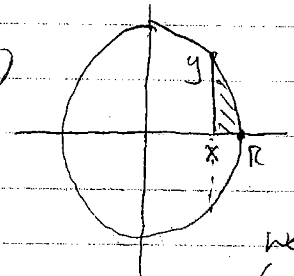
When three disk have a non-empty intersection, we say the power center of the three is the point in $\mathbb{R}^2$ with equal power distance to all three centers



III. Chord Length and Area of intersection

Claim I: Given two disks with centers, c_1 and c_2, and $D_1 \cap D_2 \neq \emptyset$, then the instantaneous change in area $\frac{d\mu}{dt}(D_1 \cap D_2) =$ the length of the mutual chord of D_1 and D_3.

(i)



We see that $y = \sqrt{r^2 - x^2}$

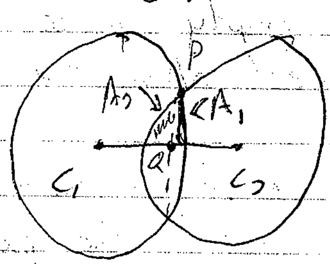
Define $f(x) = y$

To find the area A of the shaded portion we take $A = \int_x^r f(x)\,dx = -\int_x^r f(x)\,dx$

So the instantaneous rate of change of area

$$\frac{dA}{dx} = -\frac{d}{dx}\left[\int_x^r f(x)\,dx\right] = -f(x) = -y$$ by Fundamental Theorem of calculus

(ii)



Now suppose we have two disks whose distance is decreasing by a unit length per unit time, we normalize this by saying $\frac{dx}{dt} = -1$ where $x = |c_1 c_2|$

Define: $Q = c_1 c_2 \cap$ (mutual chord), $P =$ point on $\partial D_1 \cap \partial D_2$ (∂ = Boundary of), $x_1 = |c_1 Q|$, $x_2 = |c_2 Q|$, A_1 and A_2 are the shaded areas of the above figure, let $A = A_1 + A_2$, $y = |PQ|$

So, $\dfrac{dA}{dt} = \dfrac{dA_1}{dt} + \dfrac{dA_2}{dt}$

$$= \frac{dA_1}{dx_1} \cdot \frac{dx_1}{dt} + \frac{dA_2}{dx_2} \cdot \frac{dx_2}{dt}$$

$$= -y\left(\frac{dx_1}{dt} + \frac{dx_2}{dt}\right)$$

$$= -y\left(\frac{dx}{dt}\right)$$

$$= -y(-1) = y$$

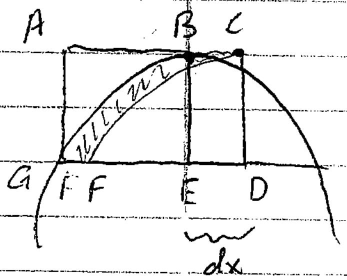
Since $\mu(D_1 \cap D_2) = 2A$, it has derivative $2y$ and $2y$ is the length of the mutual chord of D_1 and D_2.

Thus, claim 1.

Corrolary: $-\dfrac{d\mu}{dt}(D_1 \cup D_2)$ = length of the mutual chord

because the amount of area lost or gained by the intersection is by the union. (i.e. if the area of the intersection increases, then the area of the union decreases by the same amount and vice versa.)

IV. Area displaced by a circle, chord, and a line perpendicular to the chord



We want to show that if we have a circle and we displace it a distance dx, that the area swept out by rectangle $BCED = GBCF$

Notice: (i) $ACDG = ABG + GBE + BCDE$

(ii) $ACDG = ABCFG + FCD$

But $ABCFG = ABG + GBCF$

and $FCD = GBE$

By substitution into (ii), $ACDG = ABG + GBCF + GBE$

Subtracting (ii) from (i)

$$\begin{aligned} ACDG &= ABG + GBE + BCDE \\ -ACDG &= -ABG - GBE \qquad - GBCF \\ 0 &= BCDE - GBCF \end{aligned}$$

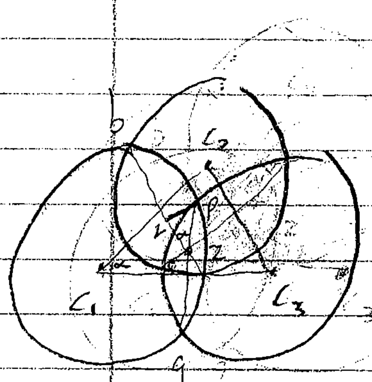
V. Claim: (Bern + Sahai)

If $D_1 \cap D_2 \cap D_3 \neq \emptyset$ and no disk contains the intersection of the other two, let $p, q \in \partial D_1 \cap \partial D_3$ inside and outside of D_2 respectively, let z be the power center of D_1, D_2, D_3 and $\mathcal{U} = \bigcup_{i=1}^{3} D_i$. Then

$$-\frac{d\mu(\mathcal{U})}{dt} = |zq| \quad \text{~~when } |c_1c_2| \text{ and } |c_2c_3| \text{ are fixed~~}$$

let $\angle c_2c_1c_3 = \alpha$

let b be the mutual chord of D_1 and D_2 and let v be the point on b where a line perpendicular to b intersects p.



Since $|c_1c_2|$ and $|c_2c_3|$ are rigid, D_1 moves perpendicular to c_1c_3 at a distance dx. So, the gain in area of

$$\frac{d\mu}{dt} D_2 \setminus (D_1 \cup D_3) = |vp| \frac{dx}{dt} = |vp| \cdot (\text{speed of } D_1 \text{ with respect to } c_1c_3)$$

Notice: $\angle PZV = \alpha$ because pq is perpendicular to c_1c_3 and vp is perpendicular to b

So $\sin\alpha = \frac{|vp|}{|zp|}$

Since D_1 is moving at $\frac{1}{\sin\alpha}$ with respect to c_1c_3 shrinking

$$\frac{d\mu}{dt}(D_2 \setminus (D_1 \cup D_3)) = \frac{|vp|}{\sin\alpha} = |vp|\left(\frac{|zp|}{|vp|}\right) = |zp|$$

Since $\mu(\mathcal{U}) = \mu(D_2 \setminus (D_1 \cup D_3)) + \mu(D_1 \cup D_3)$

$$\frac{d\mu}{dt}(\mathcal{U}) = |zp| - |pq|$$

$$= -|zq|$$

So $$-\frac{d\mu(\mathcal{U})}{dt} = |zq|$$

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|Problem Statement|

When congruent disks are pushed closer together, can the area of their union increase?

[explain]

shorten

Let $D = \{D_1, D_2, \ldots D_n\}$ be a set of overlapping balls in $\mathbb{R}^d$ w/ centers $c_1, c_2, \ldots c_n$ & radii $r_1, r_2, \ldots r_n$. Let $c_1', c_2', \ldots c_n'$ be pts. such that $|c_i' c_j'| \leq |c_i c_j|$ for each choice of i & j, where $|pq|$ is distance between p & q. $D' = \{D_1', D_2', \ldots D_n'\}$ where D' represents balls w/ center c_i' & radius r_i.

Must the d-dimensional volume of the union $U = \bigcup_{i=1}^{n} D_i$ be at least the volume of $U' = \bigcup_{i=1}^{n} D_i'$?

1st posed by Kneser & Poulsen, ('55) & ('54)

~~Union of Disks Approximatio~~

We are considering this above in 2 dimensional plane.

~~AND we~~

The opposite has been proven by Bezdek & Connolley (2001)

Namely - if a finite set of disks in the plane is rearranged so that the distance between each pair of centers does not decrease, then the area of the union does not decrease, and the area of the intersection does not increase. (in the planar case)

nope (later maybe)

We are specifically considering this in the CONTINUOUS SHRINKING condition.

Marshall Bern & Amit Sahai answered this question in 1996, which is our main topic.

(quickly)

1^{st} need to discuss area of the union of disks

Defining

$\mu(\cup_i D_i) = \mu(\cup_{i=1}^n D_i)$ where $D_1 \dots D_n$ are disks in the plane

call $\mu(\cup_i D_i) = A_1$

1^{st} Approximation of A_1 is simply

S_1 = area of individual disks $= \sum_i \mu(D_i)$

[this is unchanged by moving of disks]

Next Approximation A_2 — subtract area of overlapping two-at-a time disk intersections

$$S_2 = \cancel{\sum_{i<j} \mu(\cup_i D_i)} \sum_{i<j} \mu(D_i \cap D_j)$$

So $A_2 = A_1 - S_2 = S_1 - S_2$

Now, another, next approximation (A_3)

must add back where 3 disks intersect

$$S_3 = \sum_{i<j<k} \mu(D_i \cap D_j \cap D_k)$$

so $A_3 = S_1 - S_2 + S_3$

Final Area $= \mu(\bigcup_{i=1}^n D_i) = S_1 - S_2 + \dots + (-1)^{n-2} S_{n-1} + (-1)^{n-1} S_n$

where $S_k = \sum_{1 \le i_1 \le i_2 < \dots i_k < n} \mu(D_{i_1} \cap \dots D_{i_k})$

For 3 Disks

[Leave Eqn. at Top]

$$\mu(D_1 \cup D_2 \cup D_3) = \mu(D_1) + \mu(D_2) + \mu(D_3) - \mu(D_1 \cap D_3) - \mu(D_2 \cap D_3) - \mu(D_1 \cap D_2) + \mu(D_1 \cap D_2 \cap D_3)$$

[Zak] →

Define Problem for 3 Disks

D_1, D_2, D_3, centers c_1, c_2, c_3 & union U

Disks are moving w/ time t; μ denotes area

Suppose lengths c_1c_3 & c_2c_3 are fixed
while $|c_1c_3|$ is decreasing w/ time, normalized to $\frac{d|c_1c_3|}{dt} = -1$

Lemma - (either unchanging or decreasing at all times)

1. If $D_1 \cap D_3$ is empty or is contained in D_2, then $\mu(U)$ is unchanging.

2. Otherwise, if $D_1 \cap D_2 \cap D_3$ is empty or either D_1 or D_3 contains the intersection of the other two disks, then $-d\mu(U)/dt$ equals the length of the mutual chord of D_1 & D_3

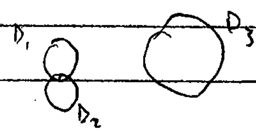
1. $\mu(U) = \mu(D_1) + \mu(D_2) + \mu(D_3) - \mu(D_1 \cap D_2) - \mu(D_1 \cap D_3) - \mu(D_2 \cap D_3) + \mu(D_1 \cap D_2 \cap D_3)$

Only terms changing w/ time are $\mu(D_1 \cap D_3)$ & $\mu(D_1 \cap D_2 \cap D_3)$

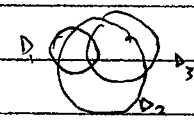
a) - if $D_1 \cap D_3$ is empty, so ~~...~~ both μ's are zero $\Rightarrow$ unchanging

b) - if $D_1 \cap D_3$ is in D_2, then the changes cancel each other out

a)



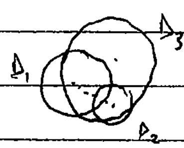
b)



2. show that $\mu(D_1 \cap D_2 \cap D_3)$ is unchanging

if empty ✓

if D_1 or D_3 contains intersection of other 2



$c_1 c_2$ is fixed, so $\mu(D_1 \cap D_2 \cap D_3)$ will not change as long as $\mu(D_1 \cap D_2)$ remains in D_3

$\Rightarrow \mu(D_1 \cap D_2 \cap D_3)$ is unchanging

~~$\mu(U) = \mu(D_1) + \mu(D_2) + \mu(D_3) + \mu(D_1 \cap D_2) + \mu(D_1 \cap D_3)$~~ +

$\mu(U) = \mu(D_1) + \mu(D_2) + \mu(D_3) - \mu(D_1 \cap D_2) - \mu(D_1 \cap D_3) - \mu(D_2 \cap D_3) + \mu(D_1 \cap D_2 \cap D_3)$

Now consider $\mu(D_1 \cap D_3)$

by Zak's explanation earlier

$\cancel{\frac{d\mu(U)}{dt}} \quad \frac{dA}{dt} = \frac{d\mu(D_1 \cap D_3)}{dt}$ = length of mutual chord

so

$-\frac{d\mu(U)}{dt}$ = length of mutual chord = $\frac{d\mu(D_1 \cap D_3)}{dt}$

& is thus decreasing