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INSCRIBED SQUARES AND SQUARE-LIKE QUADRILATERALS IN CLOSED CURVES

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Abstract. We show that for every smooth curve in $\mathbf{R}^n$, there is a quadrilateral with equal sides and equal diagonals whose vertices lie on the curve. In the case of a smooth plane curve, this implies that the curve admits an inscribed square, strengthening a theorem of Schnirelmann and Guggenheimer. "Smooth" means having a continuously turning tangent. We give a weaker condition which is still sufficient for the existence of an inscribed square in a plane curve, and which is satisfied if the curve is convex, if it is a polygon, or (with certain restrictions) if it is piecewise of class C^1. For other curves, the question remains open.

1. *Introduction.* A *simple closed curve* is a continuous function $\omega:\mathbf{R}\to\mathbf{R}^n$ that satisfies $\omega(x)=\omega(y)$, if, and only if, $x-y$ is an integer. Such a curve is determined by its values in $[0,1]$. In this paper, "curve" means simple closed curve. The curve ω is *smooth* if it has a continuous non-vanishing derivative; any curve with a continuously turning tangent can be parameterized so as to be smooth in this sense. A quadrilateral is *inscribed* in ω if all of its vertices lie on ω (that is, they are in the image im ω). In the case of a curve in $\mathbf{R}^2$, a quadrilateral may be "inscribed" even if its sides do not lie in the interior of the curve.

In this article we will show that every smooth curve in $\mathbf{R}^n$ admits an inscribed quadrilateral whose sides are all equal and whose diagonals are equal (Theorem 1, Section 4). The quadrilateral may be chosen so that its vertices have the same cyclic order in the quadrilateral as in the curve. In Section 5, the theorem is extended to curves meeting a milder smoothness condition, called "Condition A": chords which are sufficiently short and close together do not have perpendicular directions.

We are mainly interested in curves in $\mathbf{R}^2$, in which Theorem 1 implies that every smooth curve admits an inscribed square. In Section 6 this result is extended to "locally monotone" curves in $\mathbf{R}^2$ (Theorem 3). A curve is locally monotone if every point a of the curve has a neighborhood $U(a)$ and a direction $n(a)$ such that no chord of the curve is contained in $U(a)$ and parallel to $n(a)$. Smooth curves meet this condition; so do convex curves, all polygons, and most piecewise C^1 curves (see Section 6). For curves which do not satisfy this condition, the question of whether there must be an inscribed square remains open.

§1. *Previous results.* The problem of finding inscribed squares in plane curves is a very old one. Klee gives a history of the problem in [1] and [2], tracing the first proof of the existence of inscribed squares in convex polygons and convex smooth curves to Emch, in 1913 [3] and 1915 [4]. Klee also cites more general results by Schnirelmann [5] and Guggenheimer [6] (for C^1 curves) and by Jerrard [7] (for analytic curves).

Schnirelmann's paper was first published in 1929. As strengthened by Guggenheimer, Schnirelmann's result was the following: "*On every simple closed C^1 curve of declension of bounded variation one can find four points which form the vertices of a square*". The declension condition (a slightly weaker hypothesis than C^2) could be satisfied piecewise, but the C^1 condition had to be global.

Jerrard examined the case of real-analytic curves (curves whose coordinate functions are each real-analytic). He showed that generically, each such curve admits an *odd* number of inscribed squares. (In special cases, some squares must be counted with multiplicities.) Although Jerrard stated that the theorem concerning the existence of an inscribed square "can be extended quite easily to any simple, plane, closed, differentiable curve", his published proof made essential use of analyticity at several points.

In [8], Fenn proved the following theorem. Let D be a bounded, convex set in $\mathbf{R}^2$, and let $f:\mathbf{R}^2\to\mathbf{R}$ be a continuous function which is non-negative in the interior of D and zero elsewhere. Let $d>0$ be fixed. Then there is a square of side d whose center is in D, such that f takes the same value at all four vertices of the square. Fenn interprets the problem in terms of making a square table stand level on a bumpy floor. If the non-trivial level curves of f are all similar to ∂D, then Fenn's problem is equivalent to the problem of finding an inscribed square in a plane curve. Fenn's proof requires the convexity of D. Meyerson proved ([11], see also [12] and [13]) that Fenn's result cannot be extended to non-convex D in exactly the form stated. Can it be extended in some other form?

Problem Starts

§2. *Definitions.* In this section we introduce some required tools. Chief among these are a simplex Q representing the set of quadrilaterals inscribed in ω; and four subsets Q_1, Q_2, Q_3, Q_4 which cover Q and whose intersection corresponds to the set of inscribed rhombuses.*

* The Oxford English Dictionary lists *rhombi* as an eighteenth-century variant form.

Denote by Q the set

$$Q=\{(x_1,x_2,x_3,x_4)\in\mathbf{R}^4 \mid 0\le x_1\le x_2\le x_3\le x_4\le 1\}.$$

This is a 4-simplex with vertices

$$v_0=(1,1,1,1),$$
$$v_1=(0,1,1,1),$$
$$v_2=(0,0,1,1),$$
$$v_3=(0,0,0,1),$$
$$v_4=(0,0,0,0),$$

and faces $F_0,\dots,F_4$ numbered so that each F_i is opposite v_i. The boldface $\mathbf{x}$ always represents an element (x_1,x_2,x_3,x_4) of Q. The simplex is illustrated in Fig. 1. As usual, illustration of a four-dimensional object requires compromises: the right edge of the figure, drawn as a line segment, is actually the 2-simplex $F_0\cap F_4$, and therefore has three vertices. The top and bottom faces in the figure are actually 3-simplices, F_4 and F_0 respectively.

Given a fixed curve ω, we associate with each point $\mathbf{x}\in Q$ an inscribed quadrilateral with vertices at $\omega(x_1), \omega(x_2), \omega(x_3), \omega(x_4)$. The correspondence is not quite one-to-one, since the points of F_4 represent the same quadrilaterals as the points of F_0. In fact, if we define $h:F_0\to F_4$ by

$$h(0,x,y,z)=(x,y,z,1),$$

then $h(\mathbf{x})$ represents the same quadrilateral as $\mathbf{x}$, but with the vertices numbered differently. Despite this ambiguity we identify the points of Q with the corresponding geometric figures and refer to points of Q as quadrilaterals.

Every point of Q corresponds to a quadrilateral whose vertices have the same cyclic order in the quadrilateral as in ω. Some of these quadrilaterals are *degenerate*, in that they have one or more sides of zero length; and some are *one-point quadrilaterals*, in which all four sides have zero length.

For each $i=1,2,3,4$ define

$$s_i(\mathbf{x})=\|\omega(x_{i+1})-\omega(x_i)\|.$$

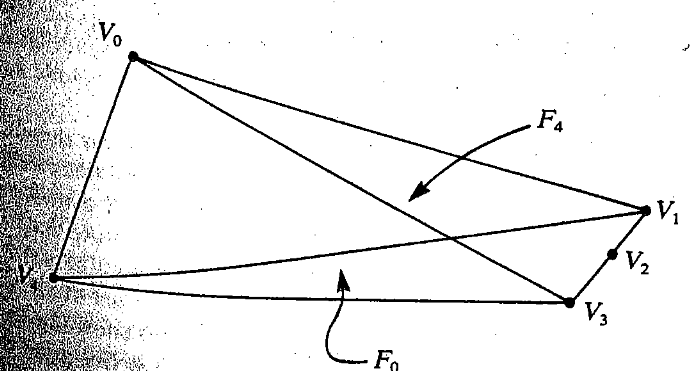

Figure 1. The simplex Q of quadrilaterals.

(When $i=4$, read x_{i+1} to mean x_1.) Thus $s_i(\mathbf{x})$ is the length of the i-th side of the quadrilateral corresponding to $\mathbf{x}$, and each s_i is a continuous function on Q.

Now for each i, define Q_i to be the closure of the set

$$\{x \in Q^0 \mid s_i(\mathbf{x}) = \max_j s_j(\mathbf{x})\}.$$

Here Q^0 denotes the interior of Q. Now Q_i is the set of quadrilaterals whose i-th side is their longest side; except that a quadrilateral in ∂Q is included in Q_i only if it is the limit of quadrilaterals in Q^0 with the same property. Each Q_i is still closed, and we still have $\bigcup_i Q_i = Q$. The purpose of this device is to prevent one-point quadrilaterals from automatically being elements of every Q_i; but the device works perfectly only if the curve is sufficiently smooth.

LEMMA 1. *If ω is a smooth curve, then each one-point quadrilateral is contained in only one set Q_i. In particular, $v_i \in Q_i$ for $i=1,2,3$ and $\mathbf{x} \in Q_4$ for each $\mathbf{x}$ on the edge connecting v_0 and v_4.*

not necessary

This lemma follows from the stronger version (Lemma 5) which will be proved in Section 5. The smoothness of ω is required only for this lemma. For what follows, assume that ω is any curve for which the result of Lemma 1 is valid.

For $i=1,2,3$ notice that Q_i includes v_i, but does not intersect the opposite face F_i (where $s_i(\mathbf{x})=0$). The case of $i=4$ is different: Q_4 includes the entire segment from v_0 to v_4, and avoids the opposite 2-simplex $F_0 \cap F_4$.

Now let $R = \bigcap_i Q_i$. We call a point $\mathbf{x} \in R$ a *rhombus*; it corresponds to an inscribed quadrilateral whose sides are equal and nonzero. A *square-like quadrilateral* is a rhombus which satisfies

$$d_{13}(\mathbf{x}) = d_{24}(\mathbf{x}),$$

where

$$d_{13}(\mathbf{x}) = \|\omega(x_3) - \omega(x_1)\|$$

and

$$d_{24}(\mathbf{x}) = \|\omega(x_4) - \omega(x_2)\|.$$

In other words, a square-like quadrilateral is an inscribed quadrilateral with equal sides and equal diagonals. In $\mathbf{R}^2$, a square-like quadrilateral is an inscribed square. (We are using the term "rhombus" to include equilateral quadrilaterals which do not lie in a plane.)

A *thin* rhombus is one which satisfies $d_{13} \geq d_{24}$, and a *fat* rhombus is one which satisfies $d_{13} \leq d_{24}$. A rhombus which is both thin and fat is a square-like quadrilateral. Denote by R_{THIN} and R_{FAT} the subsets of R consisting of thin and fat rhombuses respectively; then $R = R_{\text{THIN}} \cup R_{\text{FAT}}$. Note that if $\mathbf{x} \in F_0$ is a rhombus, then so is $h(\mathbf{x})$; but if $\mathbf{x}$ is thin, then $h(\mathbf{x})$ is fat and vice versa.

Our strategy in the next section will be to study the set of rhombuses in Q, and especially the rhombuses on F_0 and F_4. We will show that there must be, in a sense, an *odd* number of rhombuses in F_0. If the number of thin rhombuses on F_0 is even, for example, then the number of fat rhombuses on this face must be odd. The correspondence based on the function h shows that these parities must be reversed on F_4. But we will also show that the parities cannot be reversed—in effect because the bottom face can be lifted continuously up through Q onto the top face—unless R_{THIN} and R_{FAT} intersect. To prove all of this, and to handle the case in which there are infinitely many rhombuses in F_0, we will need some machinery from homology theory.

§3. *The "degree" of a set of rhombuses.* In this section we will define, in a general context, the notion of the "degree" of a subset K of a simplex, given a cover of the simplex by closed sets. When this concept is applied in the next section, K will be a set of rhombuses, and the degree will be our way of counting the rhombuses mod 2.

Although the techniques in this section are developed from scratch, they are far from new. Some of the ideas are similar to those in, for example, A. W. Tucker's paper **[9]**. A closer precedent is Sperner's Lemma, described in [10, pp. 117–119].

All homology groups in this paper are simplicial homology groups with coefficients in $\mathbf{Z}_2$.

Let A be an n-simplex. We will temporarily use the notations v_i, F_i for the vertices and faces of A. A *cover of closed vertex neighborhoods* in A, or simply a *cover*, is a family of closed subsets $A_0, \ldots, A_n$ of A such that $v_i \in A_i$ but $F_i \cap A_i = \varnothing$ for each i, and such that $\bigcup_i A_i = A$. This cover is denoted $\{A_i\}$. We are going to show that $\bigcap A_i$ is nonempty, and that in a certain well-defined sense, it is "odd" rather than "even".

Let $\{A_i\}$ be a cover. Let K be any subset of $\bigcap A_i$ which is both open and closed relative to $\bigcap A_i$. We include the possibilities $K = \varnothing$ and $K = \bigcap A_i$. The goal of the next several paragraphs is to define the "degree" of the cover $\{A_i\}$ around K. The degree of K is an element of $\mathbf{Z}_2$; in effect it tells whether we should consider K to be even or odd.

A *reversing map* for the cover $\{A_i\}$ is a function

$$f : (A - \bigcap A_i) \longrightarrow \partial A$$

which maps each set A_i into the opposite face F_i—that is, $f(A_i) \subseteq F_i$ for each i. We show that a reversing map always exists: For each i, let $d(\mathbf{x}, A_i)$ denote the distance from $\mathbf{x}$ to A_i, and define f by

$$f(\mathbf{x}) = \sum_i \frac{d(\mathbf{x}, A_i)}{\sum_j d(\mathbf{x}, A_j)}\, v_i.$$

This formula gives $f(\mathbf{x})$ as a convex combination of the vertices v_i, so that $f(\mathbf{x}) \in A$. In fact, since the coefficient of v_i is zero whenever $\mathbf{x} \in A_i$, we actually have $f : (A - \bigcap A_i) \to \partial A$, and $f(A_i) \subseteq F_i$ for each i. Note that any two reversing maps for the same cover are homotopic, so that up to homotopy, there is only one reversing map.

Now let $L = \bigcap A_i - K$. Triangulate A finely enough that no simplex of the triangulation touches both K and L. Let Γ be the n-chain consisting of the simplices which touch K. Then the boundary of Γ represents a homology class $\gamma \in H_{n-1}(A - \bigcap A_i)$; it is the unique homology class which surrounds all of K but none of L. We denote it by γ_K.

The degree of $\{A_i\}$ around K is defined as the image $f_*(\gamma_K)$ in $H_{n-1}(\partial A)$ where f is any reversing map for $\{A_i\}$. The group $H_{n-1}(\partial A)$ can be identified with $\mathbf{Z}_2$, and we will regard the degree to be an element of $\mathbf{Z}_2$. Its value is independent of the choices made in the above construction. We will denote it by $\deg_{\{A_i\}} K$ or, when the cover is identified by context, by $\deg K$.

Suppose B is any simplex, and $f: A - \bigcap A_i \to \partial B$ is any function which maps each A_i into a different face of B. Then the degree of K could just as well have been defined as $f_*(\gamma_K)$ in $H_{n-1}(\partial B)$, since there is an isomorphism $g: B \to A$ which makes gf a reversing map, and $(gf)_*(\gamma_K) = f_*(\gamma_K)$ (both being regarded as elements of $\mathbf{Z}_2$). In this case we say that f is "isomorphic to a reversing map".

leads to lemmas

We can collect some facts about these degrees. If $K = \varnothing$, then $\gamma_K = 0$, the zero element of $H_{n-1}(A - \bigcap A_i)$; and if K is the disjoint union of open-and-closed subsets K_1 and K_2 then $\gamma_K = \gamma_{K_1} + \gamma_{K_2}$. Therefore we have the following Lemmas 2 and 3.

LEMMA 2. $\deg \varnothing = 0$.

LEMMA 3. *If* $K = K_1 \cup K_2$, *then* $\deg K = \deg K_1 + \deg K_2$.

LEMMA 4. $\deg (\bigcap A_i) = 1$.

Proof of Lemma 4. Let S^{n-1} denote the $(n-1)$-sphere, and let $g: S^{n-1} \to S^{n-1}$ be any continuous map without fixed points. Then g is homotopic to the antipodal map, a homeomorphism of the sphere; and so $g_*: H_{n-1}(S^{n-1}) \to H_{n-1}(S^{n-1})$ is the identity map. Now ∂A is homeomorphic to S^{n-1}, and any reversing map f restricts to a function $\partial A \to \partial A$ without fixed points, so f induces the identity map on $H_{n-1}(\partial A) = \mathbf{Z}_2$. Looking at ∂A itself as a representative of the homology class that generates $H_{n-1}(\partial A)$, we can conclude that $f_*(\partial A) = 1$.

But ∂A surrounds $\bigcap A_i$, and so represents $\gamma_{\cap A_i}$. Therefore $f_*(\gamma_{\cap A_i}) = 1$, which proves the lemma.

Main Part

§4. *The main result.* We are now ready to return to the problem of searching for a square-like quadrilateral in the simplex Q.

THEOREM 1. *If ω is a smooth curve, then ω admits an inscribed quadrilateral with equal sides and equal diagonals.*

Proof. Let Q, R and their subsets be as before. Suppose that there is no square-like quadrilateral in Q; then R can be written as a disjoint union $R = R_{\mathrm{THIN}} \dot\cup R_{\mathrm{FAT}}$.

The face F_0 is a simplex, and it has a cover $\{F_0 \cap Q_i\}$ of closed vertex neighborhoods. The intersection of the sets in the cover is

$$F_0 \cap R = (F_0 \cap R_{\mathrm{THIN}}) \dot\cup (F_0 \cap R_{\mathrm{FAT}}).$$

From Lemmas 3 and 4, we have

$$\deg (F_0 \cap R_{\mathrm{THIN}}) + \deg (F_0 \cap R_{\mathrm{FAT}}) = \deg (F_0 \cap R) = 1,$$

from which we infer that

$$\deg (F_0 \cap R_{\mathrm{THIN}}) \neq \deg (F_0 \cap R_{\mathrm{FAT}}). \tag{1}$$

We will obtain a contradiction by showing that each side of (1) is equal to $\deg (F_4 \cap R_{\mathrm{THIN}})$ (measured in the simplex F_4).

The map $h: F_0 \to F_4$ is not only a homeomorphism of the faces; it is also an isomorphism of the covers $\{F_0 \cap Q_i\}$ and $\{F_4 \cap Q_i\}$. Therefore,

$$\deg_{\{F_0 \cap Q_i\}} (F_0 \cap R_{\mathrm{FAT}}) = \deg_{\{F_4 \cap Q_i\}} (h(F_0 \cap R_{\mathrm{FAT}})) = \deg_{\{F_4 \cap Q_i\}} (F_4 \cap R_{\mathrm{THIN}}).$$

That takes care of the right side of (1).

Now we claim that $\deg R_{\mathrm{THIN}}$ is the same whether it is measured in F_0 or in F_4. Intuitively, this is true because the degree of R_{THIN} must change continuously (*i.e.*, remain constant) as we progress smoothly up through slices of Q from the bottom face to the top face of the simplex. Making this argument precise takes some work.

Construct $f: (Q - R) \to \partial F_0$ as follows:

$$d(\mathbf{x}, Q_i) = \text{distance from } \mathbf{x} \text{ to } Q_i;$$

$$f(\mathbf{x}) = \sum_{i=1}^{4} \frac{d(\mathbf{x}, Q_i)}{\sum_j d(\mathbf{x}, Q_j)} v_i.$$

Then f maps each Q_i into the face $F_0 \cap F_i$ of F_0. Therefore f restricted to F_0 is a reversing map for the cover $\{F_0 \cap Q_i\}$, and also f restricted to F_4 is isomorphic to a reversing map for the cover $\{F_4 \cap Q_i\}$.

Triangulate Q finely enough that no simplex touches both R_{THIN} and R_{FAT}, and let Δ be the 4-chain consisting of simplices of the triangulation which touch R_{THIN}. Now $\partial\Delta$ is a 3-chain in $(Q - R)$. Let Γ be the 3-chain consisting of those simplices in $\partial\Delta$ which are not contained in F_0 or F_4. In the simplest cases, Γ can be thought of as a tube surrounding R_{THIN}, and with its ends abutting F_0 and F_4, as in Fig. 2. (More generally, Γ may consist of many tubes and more complicated shapes.)

Now the boundary $\partial\Gamma$ is a 2-chain which must represent the zero element of $H_2(Q - R)$. Therefore $f_*(\partial\Gamma) = 0 \in H_2(F_0)$. But $\partial\Gamma$ contains two com-

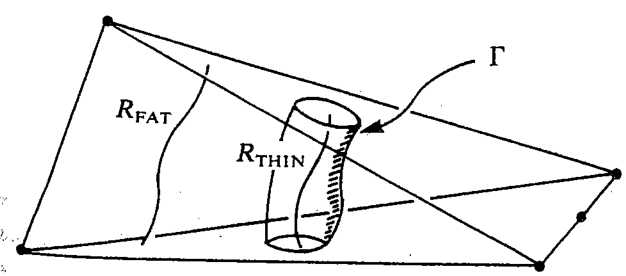

Figure 2. Simplest case of Γ.

polygons: one surrounds $(R_{\text{THIN}} \cap F_0)$ in F_0, and one surrounds $(R_{\text{THIN}} \cap F_4)$ in F_4. We have:

$$0 = f_*(\partial\Gamma) = f_*(\partial\Gamma \cap F_0) + f_*(\partial\Gamma \cap F_4);$$

so

$$f_*(\partial\Gamma \cap F_0) = f_*(\partial\Gamma \cap F_4). \tag{2}$$

But the left side of (2) measures the degree of $R_{\text{THIN}} \cap F_0$, and the right side measures the degree of $R_{\text{THIN}} \cap F_4$. Therefore these two degrees are equal, completing the promised contradiction and establishing the theorem.

Condition A"

§5. *The smoothness requirement.* The smoothness of ω has been required only for Lemma 1. In this section we give a weaker hypothesis, "Condition A", which is sufficient for Lemma 1 and therefore for the existence of an inscribed square or square-like quadrilateral. Smooth curves satisfy this hypothesis, and so do polygons with only obtuse angles (corners where the curve changes direction by less than 90°).

A *chord* is a line segment joining two points of ω.

DEFINITION. *A curve ω satisfies Condition A if each point $\omega(y)$ of the curve has a neighbourhood $U(y)$ in $\mathbf{R}^n$ such that no two chords in $U(y)$ are perpendicular.*

This definition is purely geometric; that is, it depends only on the image im ω and not on the parameterization. An equivalent definition, intuitively, is that each point of the curve has a neighbourhood (in the curve) in which any two chords (oriented in the direction of the curve) differ in direction by less than 90°. More precisely, we have the following: if ω satisfies Condition A, then each $y \in \mathbf{R}$ has a neighborhood $(y-\mu, y+\mu)$ such that, if $x_1, x_2, x_3, x_4 \in (y-\mu, y+\mu)$ with $x_1 < x_2$ and $x_3 < x_4$, then

$$(\omega(x_2) - \omega(x_1)) \cdot (\omega(x_4) - \omega(x_3)) > 0.$$

The periodicity of ω and the compactness of $[0, 1]$ insure that μ can be chosen independently of y.

LEMMA 5. *If ω satisfies Condition A, then each one-point quadrilateral in Q is contained in exactly one set Q_i. In particular: $v_i \in Q_i$ for $i = 1, 2, 3$ and $\mathbf{y} \in Q_4$ for each $\mathbf{y}$ on the edge connecting v_0 and v_4.*

Proof. Let $\mathbf{y} = (y, y, y, y)$ be a one-point quadrilateral on the edge from v_0 to v_4. We shall show that $\mathbf{y}$ has a neighborhood in Q such that, for any $\mathbf{x}$ in the neighborhood which is also in Q^0, the fourth side of $\mathbf{x}$ is the unique longest side. This will imply that $\mathbf{y}$ is in Q_4, but not in any other Q_i.

Let μ be as above. The required neighborhood of $\mathbf{y}$ consists of those elements $\mathbf{x} \in Q$ whose coordinates $x_1, \ldots, x_4$ are in $(y-\mu, y+\mu)$. Let $\mathbf{x}$ be an element of this neighborhood which is also in Q^0; so we have $0 < x_1 < \ldots < x_4 < 1$. Let z_1, z_2, z_3, z_4 be vectors in $\mathbf{R}^n$ representing the sides of the quadrilateral $\mathbf{x}$: specifically, $z_i = \omega(x_{i+1}) - \omega(x_i)$ for $i = 1, 2, 3$ and $z_4 = \omega(x_4) - \omega(x_1)$. We show that z_4 is longer than, for example, z_2. We have

$$z_4 = z_1 + z_2 + z_3,$$

so

$$z_4 \cdot z_2 = z_1 \cdot z_2 + z_2 \cdot z_2 + z_3 \cdot z_2,$$

and since all of these dot products are positive,

$$z_4 \cdot z_2 > z_2 \cdot z_2.$$

But if z_4 has a larger component in the direction of z_2 than z_2 itself does, the fourth side must be strictly longer than the second side. Similarly the fourth side is longer than the first and third sides. Since the fourth side is the unique longest side, $\mathbf{x}$ is in Q_4 and no other Q_i's, and since this is true for every $\mathbf{x} \in Q^0$ sufficiently near $\mathbf{y}$, it is also true of $\mathbf{y}$ itself.

This proof, taken literally, works for v_0 and v_4. The cases of v_1, v_2, and v_3 require more delicacy in their statement, but are essentially similar.

Lemma 5 implies the following improvement of Theorem 1.

THEOREM 2. *If ω satisfies Condition A, then ω admits an inscribed quadrilateral with equal sides and equal diagonals.*

§6. *Locally monotone curves in $\mathbf{R}^2$.* All curves in this section are in $\mathbf{R}^2$. In this section we define a much less restrictive smoothness condition, "local monotonicity", which still guarantees the existence of an inscribed square. Smooth curves, convex curves, polygons, and most piecewise C^1 curves satisfy this condition.

A *segment* of a curve ω corresponding to an interval (a, b) is the restriction of the function to that interval, $\omega|(a, b)$. We call $(b-a)$ the *length* of the segment; note that length is measured in parameter space and not in $\mathbf{R}^2$. The segment is *monotone in the direction u* (where u is a non-zero vector in $\mathbf{R}^2$) if the dot product $\omega(x) \cdot u$ is a strictly increasing function of x for $x \in (a, b)$.

No square inscribed in ω—at least, not one with its vertices in the same cyclic order in the square as in the curve—can be inscribed in a monotone segment of ω.

The curve ω is *locally monotone* if, for every $y \in \mathbf{R}$, there is an interval $(y-\mu, y+\mu)$ and a direction $u(y)$ such that $\omega|(y-\mu, y+\mu)$ is monotone in the direction of $u(y)$. If ω has this property, then the periodicity of ω and the compactness of $[0, 1]$ allow us to choose the number μ to be a constant, independent of y. In this case every segment of ω with length at most 2μ is monotone in some direction, and we say that ω is *locally monotone with constant μ*.

Locally monotonicity is a purely geometric condition; that is, it depends only on the image im ω, and not on the parameterization. Here is an equivalent definition that emphasizes the geometric nature of the condition: a curve ω is locally monotone if for every point $\omega(y)$ of the curve, there are a neighbourhood $U(y)$ in $\mathbf{R}^2$ and a direction $n(y)$ such that no chord of the curve is contained in $U(y)$ and parallel to $n(y)$. The direction $n(y)$ is normal to $u(y)$ and can be thought of as a kind of a normal vector (even if the curve is not differentiable).

To see that smooth curves are locally monotone, let $u(y)=\omega'(y)$ or take $n(y)$ to be a normal vector. To see that convex curves are locally monotone, take $n(y)$ to be a vector from $\omega(y)$ toward any interior point.

Let us make precise which piecewise C^1 curves are locally monotone. We say that a curve is piecewise C^1 if there exist numbers $x_0, \ldots, x_k$ satisfying $0=x_0<\ldots<x_k=1$ such that ω has a continuous non-vanishing derivative ω' on each interval $[x_{i-1}, x_i]$, including the one-sided derivatives at the endpoints. We denote the one-sided derivatives at x_i by $\omega'_-(x_i)$ and $\omega'_+(x_i)$. The curve has a *cusp* at x_i if these two vectors point in opposite directions; that is, if $\omega'_+(x_i)$ is a negative multiple of $\omega'_-(x_i)$. Then it can be shown that *a piecewise C^1 curve without cusps is locally monotone.* For the direction $u(x_i)$, take any convex combination of $\omega'_-(x_i)$ and $\omega'_+(x_i)$.

We are now ready to extend the main result to locally monotone curves in $\mathbf{R}^2$.

THEOREM 3. *If ω is a locally monotone curve in $\mathbf{R}^2$, then ω admits an inscribed square.*

Proof. Assume that ω is locally monotone with constant μ.

Our strategy in this proof will be to approximate ω using smooth curves ω_ε, each of which contains an inscribed square by Theorem 1. As $\varepsilon \to 0$, a subsequence of these inscribed squares converge to a square inscribed in ω. The only difficulty is in proving that the limiting square does not have side zero. To show this we will first show that each ω_ε is itself locally monotone with constant at least $\frac{1}{2}\mu$; and this will allow us to establish a lower bound on the size of the square we find in ω_ε.

For this proof, we define the *size* $\|\mathbf{x}\|$ of an element $\mathbf{x}=(x_1, x_2, x_3, x_4) \in Q$ to be the smallest of these four numbers: (x_4-x_1), $((1+x_3)-x_4)$, $((1+x_2)-x_3)$, and $((1+x_1)-x_2)$. Thus, $\|\mathbf{x}\|$ is the same as the length of the smallest segment of the curve that can contain all of the points $\omega(x_1), \omega(x_2), \omega(x_3), \omega(x_4)$. Note that $\|\mathbf{x}\|$ is measured in parameter space, and is not directly related to the lengths of the sides of the corresponding quadrilateral or any other measurement in $\mathbf{R}^2$. Nevertheless, the only quadrilaterals with size zero are the one-point quadrilaterals, and since $\|\mathbf{x}\|$ is continuous on Q, any sequence of quadrilaterals whose sizes have a positive lower bound cannot converge to a one-point quadrilateral.

Now let $\varepsilon>0$, and choose $\delta>0$ such that $|x-y|<\delta$ implies that $\|\omega(x)-\omega(y)\|<\varepsilon$. In any case choose $\delta<\frac{1}{2}\mu$. Define $\omega_\varepsilon: \mathbf{R} \to \mathbf{R}^2$ by

$$\omega_\varepsilon(x)=\frac{1}{\delta} \int_{t=0}^{\delta} \omega(x+t)\, dt.$$

Then ω_ε satisfies $\|\omega_\varepsilon(x)-\omega(x)\|<\varepsilon$ for all x, and ω_ε has a continuous non-vanishing derivative given by

$$\omega'_\varepsilon(x)=\frac{1}{\delta}(\omega(x+\delta)-\omega(x)).$$

It is not quite trivial to show that ω_ε is actually a simple closed curve; but neither is it especially difficult for sufficiently small ε, using the local monotonicity of ω. We may therefore apply Theorem 1 to find a square S_ε inscribed in ω_ε, whose vertices are in the same cyclic order in the square as in the curve.

We show that ω_ε is locally monotone with constant $\frac{1}{2}\mu$: Let $y \in \mathbf{R}$, and let $u(y)$ be chosen such that $\omega \mid (y-\mu, y+\mu)$ is monotone in the direction $u(y)$. Let x_1, x_2 be contained in $(y-\frac{1}{2}\mu, y+\frac{1}{2}\mu)$, with $x_1<x_2$. Then

$$(\omega_\varepsilon(x_2)-\omega_\varepsilon(x_1)) \cdot u(y)=\frac{1}{\delta} \int_{t=0}^{\delta} (\omega(x_2+t)-\omega(x_1+t))\, dt>0,$$

because the monotonicity of ω forces the integrand to be strictly positive. Therefore the chord from $\omega_\varepsilon(x_1)$ to $\omega_\varepsilon(x_2)$ has a positive component in the direction of $u(y)$, as required.

It follows that the inscribed square S_ε cannot have size less than μ, or its vertices would be contained in some interval of length μ, in which ω_ε would have to be monotone. We repeat this construction for a sequence of values of ε approaching zero. Some subsequence of the squares S_ε will converge to a square S, which must have size at least μ and must be inscribed in ω. This completes the proof of Theorem 3.

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References

1. Victor Klee. Some unsolved problems in plane geometry. *Math. Magazine*, 52 (1979), 131–145.
2. Victor Klee and Stan Wagon. *New and old unsolved problems in plane geometry and number theory* (Mathematical Association of America, to appear).
3. A. Emch. Some properties of closed convex curves in a plane. *Amer. J. of Math.*, 35 (1913), 407–412.
4. A. Emch. On the medians of a closed convex polygon. *Amer. J. of Math.*, 37 (1915), 19–28.
5. L. G. Schnirelmann. O nekotoryh geometriceskih svoistvah zamknutyh krivyh (On certain geometrical properties of closed curves). *Sbornik rabot matematičeskogo razdela sekcii estestvennyh i točnyh nauk Komakademii, Moskva* (1929); *Uspehi Matematičeskih Nauk*, 10 (1944), 34–44.
6. H. Guggenheimer. Finite sets on curves and surfaces. *Isreal J. Math.*, 3 (1965), 104–112.
7. R. P. Jerrard. Inscribed squares in plane curves. *Trans. Amer. Math. Soc.*, 98 (1961), 234–241.
8. Roger Fenn. The table theorem. *Bull. London Math. Soc.*, 2 (1970), 73–76.
9. A. W. Tucker. Some topological properties of disk and sphere. In *Proc. First Canadian Mathematical Congress, Montreal*, 1945 (Univ. Toronto Press, 1946), pp. 285–309.
10. S. Lefschetz. *Introduction to Topology* (Princeton University Press, 1949).
11. M. Meyerson. Convexity and the table theorem. *Pacific J. Math.*, 97 (1981), 167–169.
12. M. Meyerson. Remarks on Fenn's "The Table Theorem" and Zaks' "The Chair Theorem". *Pacific J. Math.*, 110 (1984), 167–169.
13. M. Meyerson. Balancing acts. *Topology Proceedings*, 6 (1981), 59–75.

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