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MATH 4513
Dr. Albert
Penrose Tiling Presentation

Penrose Tiles

Tiling is the use of two-dimensional geometric figures to cover the entire plane, or the analogous procedure of in higher dimensions. This is done by having a set of tiles (geometric figures) and possibly some set rules for joining to join them together; together, these two

? sets are employed to cover the plane without any gaps that aren't tiled.

In 1961, Wang conjectured ~~Wang's Conjecture was declared. This conjecture stated~~ that given any set of tiles that *could* tile the plane, these tiles would do so *periodically*. This seems reasonable since most tiling patterns that one would naturally think of are periodic. A

what is a "regular tiling"? regular tiling of the plane with polygons is called a **tessellation**. The most basic tessellation uses regular triangles as the tiles. However, only three regular polygons can tile the plane without leaving gaps.

First, we'll need to introduce some terminology. The **Schäfli symbol** is used in two dimensions to represent the number of regular p-gons which meet at a single vertex. The symbol {p} denotes a regular p-gon for integer p. The Schäfli symbol {p,q} denotes a tessellation of p-gons, with q of them surrounding a vertex.

Now, we'll prove that only three regular polygons can tile the plane without leaving gaps. Assume a regular p-gon tiles the plane with q p-gons meeting at each vertex. We have (1 – 2/p) (pi) = (2(pi))/q. This is because ((1 – 2/p) (pi)) is the value of one interior angle of a regular ~~the~~ p-gon ~~which is determined by the number of sides of one regular polygon~~ which must equal ~~and because~~ ((2(pi))/q) ~~is the value of one angle of the p-gon which is determined by the~~

because there are q

~~number of~~ p-gons touching at one vertex. ~~These must be equal because the p-gons all intersect at one vertex.~~ So, (1 – 2/p) (pi) = (2(pi))/q ➔ (pi) – (2(pi))/p = (2(pi))/q ➔ 1/p +1/q = ½ ➔ (p +q)/(pq) = ½ ➔ (½)(pq) = p + q ➔ pq - 2p – 2q = 0 = 4 – 4 ➔ pq – 2p – 2q + 4 = 4 ➔ (p – 2)(q – 2) = 4. ~~This factorization yields~~: Thus there are three possibilities for p and q, corresponding to the three following factorizations of 4:

4 = 4 * 1 = (6 – 2)(3 – 2) ➔ p = 6, q = 3 ➔ {6,3}

4 = 2 * 2 = (4 – 2)(4 – 2) ➔ p = 4, q = 4 ➔ {4,4}

4 = 1 * 4 = (3 – 2)(6 – 2) ➔ p = 3, q = 6 ➔ {3,6}

q

what do you mean by this?

Thus, only {6,3}, {4,4}, and {3,6} yield integer results, which implies that {6}, {4}, and {3} are the only regular p-gons which tile the plane. Thus, regular triangles, squares, and regular hexagons are the only polygons which tile the plane without leaving gaps.

Since regular polygons with 3, 4, or 6 sides tile the plane, it seems unintuitive at first that the regular pentagon will not tile the plane without leaving gaps. Also, if Wang's Conjecture is true, it seems reasonable to think that the pentagon will periodically tile the plane, as the tessellation {6,3}, {4,4}, and {3,6} do. However, if you try to tile the plane using regular pentagons, you notice that gaps are produced which are smaller than the pentagon being used; i.e. the gaps can't be filled with the tiles. This makes it seem that tiling the only with regular pentagons is impossible. By Wang's Conjecture, it seems that one might be able to tile the plane periodically using regular pentagons and some other polygon(s).

This might be stretching the point a bit: the truth of Wang's conjecture seems to be independent of the question of which polygons tile the plane

Yes, and of course your argument above is a pretty intuitive demonstration of why it is impossible.

again, This is a question which is independent of Wang's conjecture.

Although it won't be proved in this paper, Robert Berger disproved Wang's Conjecture. He found a set of 20426 tiles that tiled the plane **aperiodically**. By an aperiodic tiling, we mean a tiling consisting of a set of geometric tiles which tile the plane in such a way that there isn't exact translational symmetry; i.e. we mean a set of

Give a reference!

tiles which tile the plane, but they don't do so periodically. In 1966, Berger also proved that there is no fixed algorithm to determine if a given set of arbitrary tiles will tile the plane. This is because tiling problems lead to Diophantine equations (which also won't be proved). Berger (when?) soon discovered an aperiodic set of 104 tiles. In 1970, Raphael ? Robinson discovered an aperiodic set which consisted of only six tiles. (when?) Then, Roger Penrose found an even smaller aperiodic set, consisting of just *two* tiles.

Give references

Penrose's discovery is closely related to tessellation with regular pentagons. Notice that if you try to tile the plane with regular pentagons, you are left with gaps. We proved on the homework that for a regular pentagon with side lengths one the diagonal is of length phi. Consider a regular pentagon with side lengths of phi. Working with ok similar triangles, we see that the diagonal is of length (phi)^2. Continuing to work with similar triangles (proof on back), we find the dimensions of the triangles which Penrose constructed. Penrose cut the regular pentagon along the diagonals which connect the vertices. This cuts the pentagon into ten triangles and a smaller regular pentagon. We see that five of these triangles have two sides of length 1 and one of length phi; call this triangle A. The other five triangles have two side lengths of 1 and one of length 1/(phi); call this triangle B. So far, this gives us two different kinds of triangles. Now, we notice that the triangle with two side lengths of 1 and one of length 1/(phi) is similar to the triangle with two side lengths of phi and one of length 1; call this triangle C. From these three triangles, we can construct "kites," "dart," and "diamonds." Kites are constructed by pasting two C's together such that two of the sides of length phi are touching (as shown on back). Darts are constructed by pasting two A's together such that two of the sides of length 1 are touching. Lastly, diamonds are constructed either by pasting two B's together such that sides of length 1/(phi) are touching or by pasting two A's together such that sides of length phi are touching.

Don't these two statements contradict each other?

Penrose tiles only tile the plane aperiodically. To do this, however, they set of two tiles must also be accompanied by a rule. Otherwise, periodic tilings could result. Tiles are arranged so that they do *not* form parallelograms. One form of the rule uses colors to show which kites and darts go together. (This color pattern is shown on the back page.) Like colors go together. Or, notches and spikes can be added so that tiles match together in the certain way; this is done in such a way that the notch/spike method works exactly as the color method does. They are congruent rules. There are other Penrose tiles besides the kite and dart (e.g. the diamond). These tiles have other rules used for the matching.

?

does this mean diamonds tile aperiodically?

I think you are saying here that not only do like colors go together, but they should form unbroken curves? You should make this clear.

The Penrose tiling produced is **quasi-periodic**. This means that the pattern is locally symmetric, but the whole tiling cannot be translated without. Translating the pattern might show that some patterns are symmetric, but other tiles will not line up. You may choose an arbitrarily-large bounded region and translate it such that it matches perfectly with another bounded region in the plane. By shifting the tiles, however, areas outside the bounded region will *not* match up perfectly. This is why the Penrose tiling is aperiodic; while portions are symmetric, further portions aren't.

without what?

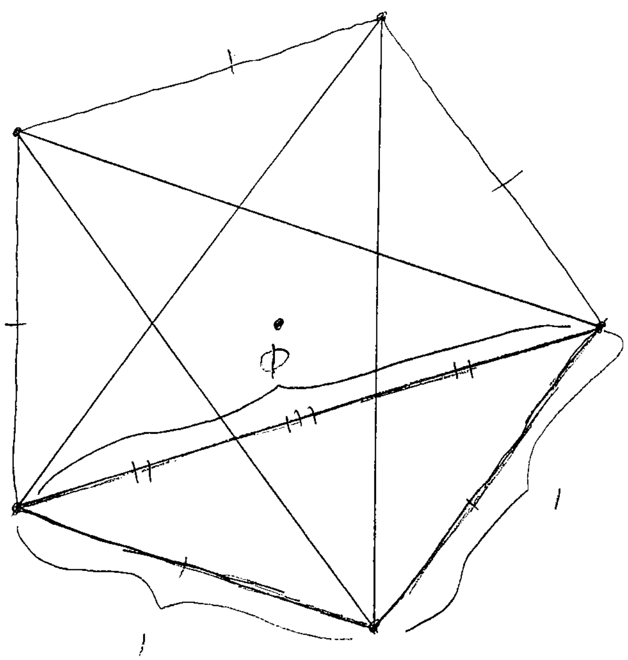
Not sure what your use of the word 'symmetric' here is.

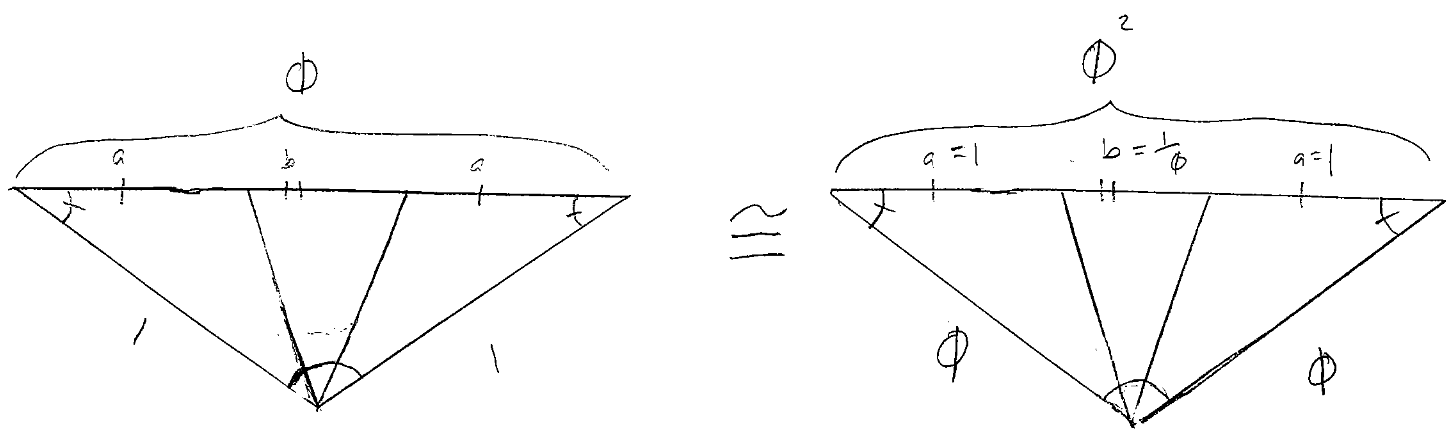
This is more clear than the previous two sentences!

Here is a good time to briefly talk about two related topics of Penrose tiling. Phi is an important number in the construction of the tiles. An alloy of aluminum-magnesium is a practical three-dimensional application.

Phi has been a perplexing number for quite a while, since it appears that this number was used in the construction of the Egyptian pyramids. Euclid discussed

From homework, we have





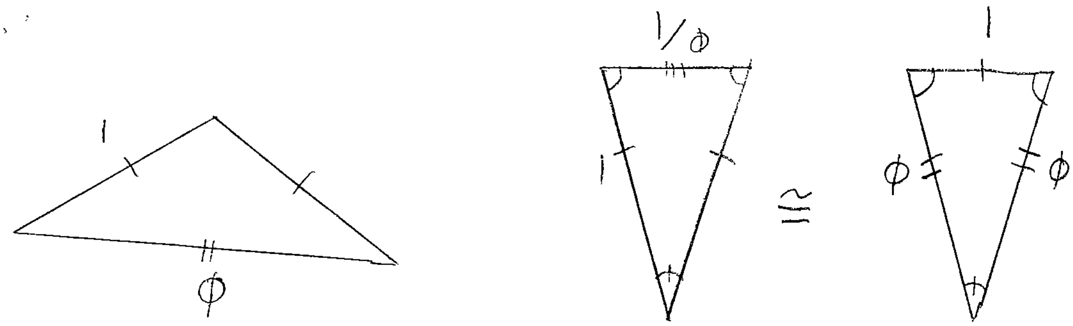
$$\phi^2 = \left(\frac{1+\sqrt{5}}{2}\right)^2 = \frac{1+2\sqrt{5}+5}{4} = \frac{6+2\sqrt{5}}{4} = \frac{3+\sqrt{5}}{2}$$

$$= \frac{6+2\sqrt{5}}{4} = \frac{8-2+2\sqrt{5}}{4} = 2 + \frac{2-2\sqrt{5}}{4}$$

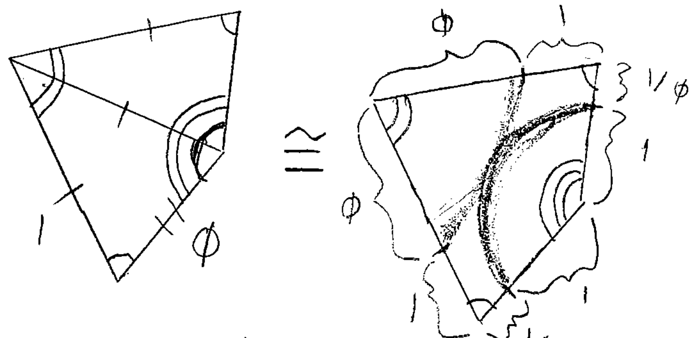
$$= 2 + \frac{2-2\sqrt{5}}{1\cdot 5} = 2 + \frac{2}{1+\sqrt{5}}\left(\frac{1-\sqrt{5}}{1-\sqrt{5}}\right) = 2 + \frac{2}{1+\sqrt{5}}$$

$$= \boxed{1+1+1/\phi = 2(a)+b \Rightarrow a=1,\ b=1/\phi.}$$

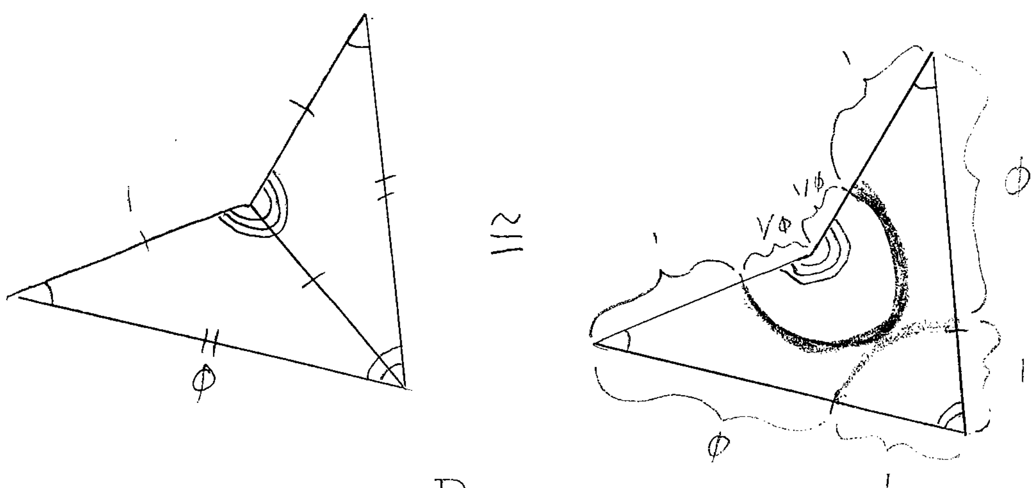
To determine a and b you need more than just the equation $2 + \frac{1}{\phi} = 2a + b$.



ΔA ΔB ΔC



Kite



Dart