Math 4513
Some unsolved problems in plane geometry

The eleven problems below are abbreviated versions of problems taken from part 1 of *Old and New Unsolved Problems in Plane Geometry and Number Theory* by Victor Klee and Stan Wagon, which I will put on reserve for this course in the mathematics library. The book of Klee and Wagon explains these problems in detail, and provides insight into them by describing how to solve some closely related problems which are not quite as hard.

Part 2 of the book describes problems on number theory, which I will summarize in an upcoming handout. There is also a part 3 containing problems on "interesting real numbers" such as π and e.

1. Think of polygons as representing rooms with mirrored walls, and assume that a ray of light which hits a wall will reflect in the usual way (i.e., the angle of incidence equals the angle of reflection). Does there exist a polygon which has a point P inside it such that a light source placed at point P will fail to illuminate the entire polygon?

2. A *chord* of a region in the plane is a line segment which starts and ends on the boundary of the region. A point P in the region is equichordal if every chord passing through P has the same length. Does there exist a convex region in the plane which has two different equichordal points? Intuition suggests that the answer is no, but no one has been able to prove this. (Note: a region is defined to be convex if every line segment joining two points of the region lies completely within the region.)

This problem has been solved; in 1997 Marek Rychlik proved that there are no convex regions in the plane with two different equichordal points. However, the topic is still a suitable one for a presentation.

3. Suppose we are given n discs, all with the same radius, at certain locations in the plane (possibly overlapping each other). Is it possible that we can move the discs to new locations which are closer together, so that the distances between the centers of the discs are all less than they were originally, and yet the total area of the union of the discs is greater than before we moved them together? (Obviously this is impossible when $n = 2$, and it seems obvious that it should be impossible for any n, but no one has yet been able to prove this.)

4. Suppose C is a convex set in the plane with the property that every plane set S with diameter equal to 1 can be fit inside C (without having to rotate S). What is the smallest that the area of such a set C can be?

5. If you draw three points in the plane and they are not all on the same line, then they form the vertices of a triangle. However, it's easy to draw four points in the plane so that no three of them are on the same line, and yet they do not form the vertices of a convex quadrilateral. (Try it.) On the other hand, it can be shown that if you draw five points in the plane and no three of them are on the same line, then you can always pick out four of them and connect those four points by line segments to obtain a convex quadrilateral. Thus five points in the plane "guarantee a convex quadrilateral", although four do not. In general, given a number n, how many points in the plane does it take to guarantee a convex n-gon in this sense?

6. Suppose you are given n points in the plane which are not all on the same line. Draw all the lines which connect two (or more) of the points. Will you always be able to find at a set of least $n/3$ of these connecting lines which all intersect in a common point?

7. Is there a polygon which tiles the plane but cannot do so periodically? ("Tiling the plane" means that if you had an infinite collection of tiles in the shape of that polygon, you could fit them together to cover the entire plane without any overlaps or spaces in between. "Tiling the plane periodically" means roughly that the polygon tiles the plane in such a way that if you were to stand above one point of the tiled plane and look down at the tiling, and then move to a certain other point and look down from there, the tiling would look exactly the same to you as it did before you moved.)

8. What is the minimum number of colors for painting the plane in such a way that no two points that are a distance 1 inch apart receive the same color?

9. Can a circle be decomposed into finitely many sets that can be rearranged to form a square? (This problem was solved as the book was going to press. [What do you guess the answer is: yes or no?] However, other similar problems remain unsolved.)

10. A set S of points in the plane is called *rational* if the distance between any two points in S is a rational number. A set S of points in the plane is called *dense* if every circle in the plane (no matter where it is and no matter how small) contains at least one element of S. Does there exist a set S of points in the plane which is both rational and dense?

11. Suppose you are given a simple closed curve in the plane. ("Simple" means the curve does not intersect itself, "closed" means the curve ends at the same point where it begins.) Can you always find four points on the curve which form the vertices of a square?