Solutions to some exercises on outer measure

#21, p. 352. We'll show that $\mu^*(E) = 1$ for every nonempty subset E of X. To see this, observe that the only collection of sets in $\mathcal{S}$ which covers E (ignoring the empty set, which has no effect), is the collection consisting of the single set X. So, by definition the outer measure of E equals $\mu(X)$, which is 1.

It is easy to verify from the definition of measurable set that both $\emptyset$ and X are measurable. (Alternatively, we could use the fact that if μ is a premeasure on an algebra $\mathcal{A}$, then every set in $\mathcal{A}$ is μ^* measureable; this was proved as part of the proof of the Carathéodory-Hahn extension theorem. In this case $\mathcal{S}$ is a σ-algebra and therefore also an algebra, and μ is clearly a measure on $\mathcal{S}$, so it is a premeasure on $\mathcal{S}$.)

If E is any subset of X which is neither empty, nor equal to all of X, then E is not measurable. To see this, let x_1 be an element in E and let x_2 be an element in E^c, and let $A = \{x_1, x_2\}$. Then A, $A \cup E$, and $A \sim E$ are all non-empty, so $\mu^*(A) = \mu^*(A \cup E) = \mu^*(A \sim E) = 1$, so $\mu^*(A) \neq \mu^*(A \cup E) + \mu^*(A \sim E)$. Hence E is not measurable.

This proves that the collection of measurable sets in X is exactly the collection $\{\emptyset, X\}$.

#25, p. 357. The only finite disjoint collections of sets $\{E_k\}_{k=1}^n$ in $\mathcal{S}$ are the collection $\{X\}$ and the collection $\{A\}$, with only one set each, so it's trivial that μ is finitely additive. To verify that μ is countably monotone, it's enough to check that $\mu(A) \leq \mu(X)$, since the only way to cover one set in $\mathcal{S}$ by a union of other sets in $\mathcal{S}$ is to take $A \subseteq A$, $A \subseteq A \cup X$, $X \subseteq A \cup X$, and $X \subseteq X$, and in all these cases the measure μ of the set on the left is less than or equal to the sum of the measures of the sets on the right. So μ is a premeasure on $\mathcal{S}$.

We can extend μ to a measure as follows. Let Σ be the collection of sets consisting of the four sets $\emptyset$, A, $X \sim A$, and X. Define a set function ν on Σ by setting $\nu(\emptyset) = 0$, $\nu(A) = 1$, $\nu(X \sim A) = 1$, and $\nu(X) = 2$. Then Σ is a σ-algebra which contains $\mathcal{S}$, ν is a measure on Σ, and ν agrees with μ on $\mathcal{S}$.

Notice, however, that ν does not agree with the outer measure μ^* induced by μ on Σ. In fact, from the definition of μ^*, we see easily that $\mu^*(E) = 2$ whenever E is a subset containing points which are not in A, $\mu^*(E) = 1$ whenever E is a nonempty subset of A, and $\mu^*(E) = 0$ when E is empty. In particular, $\mu^*(X \sim A) = 2$, so $\mu^*(X \sim A) \neq \nu(X \sim A)$.

We claim that the only μ^*-measurable sets are $\emptyset$ and X. To see this, suppose E is any set which is neither empty nor equal to all of X. There are two cases to consider: either E contains points which are not in A, or E is a subset of A. In the first case, $\mu^*(X \cap E) = \mu^*(E) = 2$ and, since $X \sim E$ is not empty, $\mu^*(X - E) \geq 1$. So

$$\mu^*(X \cap E) + \mu^*(X \sim E) = 3 \neq \mu^*(X),$$

which proves that E is not μ^*-measurable. In the second case, when $E \subseteq A$, we have $\mu^*(X \sim E) = 2$ and $\mu^*(X \cap E) = 1$, so again

$$\mu^*(X \cap E) + \mu^*(X \sim E) = 3 \neq \mu^*(X),$$

and again E is not μ^*-measurable.

#26, p. 357. The only disjoint collections of non-empty sets in $\mathcal{S}$ whose unions are also in $\mathcal{S}$ are collections with just one set, so μ is trivially finitely additive on $\mathcal{S}$. (Note that $[0,1]$ and $[2,3]$ are disjoint sets in $\mathcal{S}$, but their union is not in $\mathcal{S}$.) The only ways to cover one set in $\mathcal{S}$ by a union of nonempty sets in $\mathcal{S}$ are: $[0,1] \subset [0,3]$, $[2,3] \subset [0,3]$, and covers in which a set is covered by itself. Since $\mu([0,1]) \leq \mu([0,3])$ and $\mu([2,3]) \leq \mu([0,3])$, it follows that μ is countably monotone on $\mathcal{S}$. Hence μ is a premeasure on $\mathcal{S}$.

However, unlike the measure in problem 25 above, this premeasure μ cannot be extended to a measure ν on any σ-algebra containing $\mathcal{S}$. For any σ-algebra containing $\mathcal{S}$ must also contain the set $[0,1] \cup [2,3]$, and since $[0,1] \cup [2,3] \subseteq [0,3]$, if ν is a measure defined for these sets then we must have

$$\nu([0,1]) + \nu([2,3]) = \nu([0,1] \cup [2,3]) \leq \nu([0,3]).$$

Therefore ν could not be an extension of μ, because

$$\mu([0,1]) + \mu([2,3]) = 2 > \mu([0,3]).$$

It's not hard to see from the definition of μ^* that $\mu^*(E) = 1$ for any nonempty subset E of $\mathbf{R}$ which is contained in $[0, 3]$, and (using the convention mentioned in the footnote at the bottom of page 350) $\mu^*(E) = \infty$ for any subset E of $\mathbf{R}$ which has at least one point in common with $\mathbf{R} \sim [0, 3]$.

The μ^*-measurable sets consist of all subsets of the form $A \cup B$, where $A \subseteq \mathbf{R} \sim [0, 3]$ and either $B = \emptyset$ or $B = [0, 3]$. To see this, first let's check that any such set is measurable.

Suppose A is any subset of $\mathbf{R} \sim [0, 3]$, and C is any subset of $\mathbf{R}$. There are three possibilities: either C is empty, or $C \subset [0, 3]$, or C has has at least one point in common with $\mathbf{R} \sim [0, 3]$. In the first case, clearly

$$\mu^*(C) = \mu^*(C \cap A) + \mu^*(C \sim A)$$

is true, because all three measures are zero. In the second case, we have $\mu^*(C) = 1$, $\mu^*(C \cap A) = 0$, and $\mu^*(C \sim A) = 1$, so the above equation is still true. In the third case, we have $\mu^*(C) = \infty$, and since the sets $C \cap A$ and $C \sim A$ together cover C, at least one of them must contain a point in $\mathbf{R} \sim [0, 3]$, so one of the measures on the right side of the equation must be infinite. Therefore the equation again holds in this case. So A is measurable.

Now $B = \emptyset$ is measurable, and $B = [0, 3]$ is measurable because for every nonempty subset C of $\mathbf{R}$, if $C \subseteq [0, 3]$ then

$$1 = \mu^*(C) = \mu^*(C \cap B) + \mu^*(C \sim B) = 1 + 0,$$

and if C has at least one point in common with $[0, 3]$ then the equation still holds, by the same argument as in the preceding paragraph.

Since all sets A and B of the above form are measurable, it follows that the unions $A \cup B$ of sets of the above form are also measurable.

Now let's show that any set E not of the form $A \cup B$ with $A \subseteq \mathbf{R} \sim [0, 3]$ and $B = \emptyset$ or $B = [0, 3]$ is not measurable. If E is not of this form, then E has at least one element x_1 in common with $[0, 3]$, and there is at least one element x_2 of $[0, 3]$ which is not in E. Let $C = \{x_1, x_2\}$. Then $\mu^*(C) = 1$, $\mu^*(C \cap E) = \mu^*(\{x_1\}) = 1$, and $\mu^*(C \sim E) = \mu^*(\{x_2\}) = 1$ Hence

$$1 = \mu^*(C) \neq \mu^*(C \cap E) + \mu^*(C \sim E) = 2,$$

so E is not measurable.

#27, p. 357. Every outer measure is countably monotone, so if μ^* is an extension of μ, then because μ^* is countably monotone, then μ must be also.

Conversely, suppose μ is countably monotone. We want to prove that μ^* is an extension of μ, or in other words that for every set E in $\mathcal{S}$, $\mu^*(E) = \mu(E)$. From the definition of μ^* it is clear that $\mu^*(E) \leq \mu(E)$, so it suffices to prove the reverse inequality. For every $\epsilon > 0$, by definition of μ^* there exist sets E_k in $\mathcal{S}$ such that $E \subseteq \bigcup_{k=1}^{\infty} E_k$ and

$$\sum_{k=1}^{\infty} \mu(E_k) \leq \mu^*(E) + \epsilon.$$

Since μ is countably monotone, we have

$$\mu(E) \leq \sum_{k=1}^{\infty} \mu(E_k).$$

Combining with the preceding inequality we get

$$\mu(E) \leq \mu^*(E) + \epsilon,$$

and since ϵ is arbitrary, it follows that $\mu(E) \leq \mu^*(E)$, as desired.