Comments on problem 10, p. 130

To do this problem easily, you need to realize a fact which I briefly stated in class (but I'm not sure how explicit I made it): Suppose you have a family of extremals $y(x)$ of a functional emanating from a given point in the xy-plane; i.e., the family of all solutions of the Euler equation which satisfy $y(a) = y_0$, where a and y_0 are fixed. Let C be the envelope curve for this family of extremals; i.e. C is the curve of points where "two neighboring extremals in the family intersect each other". Then for each extremal in the family, the conjugate points to a (for this particular extremal) are the points where this extremal touches C.

(A somewhat more precise definition of the envelope is as follows. For each fixed extremal $\hat{y}$ with $\hat{y}(a) = y_0$, we will define a point P on the envelope by taking the limit of intersection points of $\hat{y}$ with neighboring extremals. To do this, take any extremal $\tilde{y}$ with $\tilde{y}(a) = \tilde{y}_0$ and $\tilde{y}'(a)$ close to $\hat{y}'(a)$, and look at a point $(\tilde{x}, \tilde{y})$ in the xy-plane where the graphs of $\hat{y}$ and $\tilde{y}$ intersect. Now change $\tilde{y}$ so that its graph comes closer and closer to the graph of $\hat{y}$; we can do this by making $\tilde{y}'(a)$ come closer and closer to $\hat{y}'(a)$. As we do this, the intersection point $(\tilde{x}, \tilde{y})$ will come closer and closer to a limiting point P. We then say that P is a point on the envelope of the family. So the envelope is, by definition, the curve composed of all the limiting points P we get in this way by starting with all possible extremals $\hat{y}$.)

Now back to the problem. Since $F = \dfrac{y}{(y')^2}$ is independent of x, the Euler equation has the first integral $y'F_{y'} - F = C$, which reduces to the equation

$$\frac{-3y}{(y')^2} = C.$$

Solving for y' and separating variables gives

$$\int \frac{y'}{\sqrt{y}}\, dx = \int P\, dx,$$

where P is a constant. Integrating both sides and solving for y gives

$$y = (Px + Q)^2,$$

where P and Q are arbitrary constants. Thus the family of extremals for J consists of parabolas opening upwards, with their vertices at arbitrary points on the x-axis and with arbitrarily large steepness. Graphing a few of the parabolas in this family makes clear that their envelope is the x-axis (not $x = 0$ as stated in the text's hint).

Checking $F_{y'y'}$ we see that $F_{y'y'} = 6y/(y')^4$, which is greater than or equal to zero for $x \in [0, a]$ for every extremal, and which is strictly positive at each x for which $y(x) \neq 0$.

Now using the conditions $y(0) = 1$ and $y(a) = A$ to find P and Q, we find that there are two possibilities for y: either

$$y = y_1 = \left[\left(\frac{1 - \sqrt{A}}{a}\right) x - 1\right]^2$$

or

$$y = y_2 = \left[\left(\frac{1 + \sqrt{A}}{a}\right) x - 1\right]^2.$$

As explained above, for each of these two extremals, the conjugate point to 0 (for that particular extremal) is the point where it touches the envelope, which in this case is the x-axis. It is easy to see that y_1 touches the x-axis at a single point, which is outside the interval $[0, a]$. So for y_1 there are no conjugate points to 0 in $[0, a]$. Also, since $y_1 \neq 0$ on $[0, a]$ then as noted above $F_{y'y'} > 0$ on $[0, a]$. Therefore y_1 satisfies the sufficient conditions for a local minimum given in class, so y_1 is a local minimum for J in $D_1[0, a]$.

For y_2, on the other hand, there is a conjugate point in $[0, a]$, so we have *almost* verified that Jacobi's necessary condition for a local minimum (stated in class and proved in the text; see p. 112) is not satisfied by y_2. This would then show that y_2 is not a local minimum for J, since it does not satisfy the necessary condition. The one little difficulty is that as a hypothesis of Jacobi's necessary condition we must assume that $F_{y'y'} > 0$ on $[0, a]$ (see text, p. 112), and that is not the case for y_2. One might consider checking the proof of Jacobi's necessary condition given in the text to see if it could still be made to work under the assumption that $F_{y'y'}$ vanishes at one point in $[0, a]$, but I haven't tried to do this.