Suppose $\{a_n\}$ is a sequence of distinct complex numbers, and $\{A_n\}$ is an arbitrary sequence of complex numbers. Show that there exists an entire function $f(z)$ such that for all $n \in \mathbf{N}$, $f(a_n) = A_n$.

Proof: By Weierstrass' theorem, there exists an entire function $g(z)$ such that $g(a_n) = 0$ and $g'(a_n) = 0$ for every $n \in \mathbf{N}$. We claim that numbers γ_n can be chosen such that

$$\sum_{n=1}^{\infty} g(z) \frac{e^{\gamma_n(z-a_n)}}{z-a_n} \frac{A_n}{g'(a_n)}$$

converges for all $z \in \mathbf{C} \backslash \{a_n\}$ to a function $f(z)$ which is holomorphic on $\mathbf{C} \backslash \{a_n\}$ and satisfies $\lim_{z \to a_n} f(z) = A_n$ for every $n \in \mathbf{N}$.

To see this, first observe that we may assume that $a_n \neq 0$ for all $n \in \mathbf{N}$. For if the problem has been solved in this case, then in the general case we can choose $b \in \mathbf{C}$ be such that $a_n + b \neq 0$ for all n. Letting $\tilde{a}_n = a_n + b$, we have that there exists an entire function $\tilde{f}$ such that $\tilde{f}(\tilde{a}_n) = A_n$ for all n. Then take $f(z) = \tilde{f}(z - b)$.

We now define $\gamma_n = q_n / a_n$, where q_n is any sequence of nonnegative real numbers such that

$$\sum_{n=1}^{\infty} \left| \frac{A_n}{g'(a_n)} \right| e^{-q_n/2} < \infty.$$

Clearly such sequences exist; for example we could define q_n to be the greater of 0 and $-2 \log \left(\frac{1}{2^n} \left| \frac{g'(a_n)}{A_n} \right| \right)$.

Let R be an arbitrary positive number. Then there exists $N \in \mathbf{N}$ such that for all $n \geq N$, $|a_n| \geq 2R$. So if $|z| \leq R$ and $n \geq N$, then $|z| \leq |a_n|/2$, and hence

$$\begin{aligned}
\text{Re } (\gamma_n(z - a_n)) &= \text{Re } (-\gamma_n a_n) + \text{Re } (\gamma_n z) \\
&= -q_n + \text{Re } (\gamma_n z) \\
&\leq -q_n + |\gamma_n z| \\
&\leq -q_n + |\gamma_n a_n|/2 \\
&= -q_n + q_n/2 = -q_n/2.
\end{aligned}$$

Therefore, for all $n \geq N$ and $|z| \leq R$, we have $\left| e^{\gamma_n(z-a_n)} \right| \leq e^{-q_n/2}$. Furthermore, $|z - a_n| \geq |a_n| - |z| \geq |a_n|/2 \geq R$.

Since g is continuous, there exists $M \geq 0$ such that $|g(z)| \leq M$ for $|z| \leq R$. So, for all $n \in \mathbf{N}$ and $|z| \leq R$,

$$\left| g(z) \frac{e^{\gamma_n(z-a_n)}}{z-a_n} \frac{A_n}{g'(a_n)} \right| \leq \frac{M}{R} e^{-q_n/2} \left| \frac{A_n}{g'(a_n)} \right|.$$

It follows from the Weierstrass M-test that the series

$$\sum_{n=N}^{\infty} g(z) \frac{e^{\gamma_n(z-a_n)}}{z-a_n} \frac{A_n}{g'(a_n)}$$

converges to a holomorphic function $f_1(z)$ on $\{|z| < R\}$.

On the other hand, the function $f_2(z)$ defined by

$$f_2(z) = \sum_{m=1}^{N-1} g(z) \frac{e^{\gamma_m(z-a_m)}}{z-a_m} \frac{A_m}{g'(a_m)}$$

is holomorphic on $\mathbf{C} \backslash \{a_1, \ldots, a_N\}$ and has a removable singularity at a_n for each $n \leq N - 1$. In fact, $\lim_{z \to a_n} f_2(z) = A_n$, as is easily seen from the fact that $g(a_n) = 0$. Therefore $f_1(z) + f_2(z)$ extends to a function $f(z)$ which is holomorphic on $\{|z| < R\}$ and satisfies $f(a_n) = A_n$ for all n such that $|a_n| < R$. Since R was arbitrary, this completes the proof.