Definition. If $A \subseteq \mathbf{R}$ and $c \in A$, we say c is an *interior point* of A if there exists a neighborhood $V_\delta(c)$ of c such that $V_\delta(c) \subseteq A$.

If I is an interval, then every point c in I is an interior point of I, unless c is an endpoint of I.

Theorem. Suppose $I \subseteq \mathbf{R}$ is an interval, $f : I \to R$, and $c \in I$. If

(i) c is an interior point of I,
(ii) f has a relative extremum at c, and
(iii) f is differentiable at c, then

$$f'(c) = 0.$$

Proof. Let $g(x) = \dfrac{f(x) - f(c)}{x - c}$ for $x \in I$, $x \neq c$. Since f is differentiable at c, then $\lim_{x \to c} g(x) = f'(c)$ exists. We will prove that $f'(c) = 0$ by showing that the assumptions $f'(c) > 0$ and $f'(c) < 0$ each lead to a contradiction.

Consider first the case in which $f'(c) > 0$. By (ii), f has either a relative maximum or relative minimum at c. Then we have two subcases to consider: either $f'(c) > 0$ and f has a relative maximum at c, or $f'(c) > 0$ and f has a relative minimum at c.

In the first subcase ($f'(c) > 0$ and f has a relative maximum at c), we get a contradiction as follows. Since $f'(c) > 0$, then $\lim_{x \to c} g(x) = f'(c) > 0$. So by Theorem 4.2.9 (proved in class), there exists a neighborhood $V_{\delta_1}(c)$ of c such that if $x \in (V_{\delta_1}(c) \cap I)$ and $x \neq c$, then $g(x) > 0$, or

$$\frac{f(x) - f(c)}{x - c} > 0. \tag{1}$$

Also, since f has a relative maximum at c, then there exists a neighborhood $V_{\delta_2}(c)$ of c such that if $x \in (V_{\delta_2}(c) \cap I)$, then

$$f(x) \leq f(c). \tag{2}$$

Finally, since c is an interior point of I, there exists a neighborhood $V_{\delta_3}(c)$ of c such that $V_{\delta_3}(c) \subseteq I$. Now let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, and choose x such that $c < x < c + \delta$. (It doesn't matter which value of x you choose in this range; for example, $x = c + \delta/2$ will do.) Then $x \in V_{\delta_3}(c)$, so $x \in I$, and also $x \in V_{\delta_1}(c)$ and $x \in V_{\delta_2}(c)$, so both the inequalities (1) and (2) displayed above hold. But since $c < x$, then

$$x - c > 0. \tag{3}$$

This gives a contradiction, because multiplying inequality (1) by inequality (3) gives $f(x) - f(c) > 0$, which contradicts inequality (2).

In the second subcase ($f'(c) > 0$ and f has a relative minimum at c), we get a contradiction as follows. As above, there exists a neighborhood $V_{\delta_1}(c)$ such that if $x \in (V_{\delta_1}(c) \cap I)$ and $x \neq c$, then $g(x) > 0$, or

$$\frac{f(x) - f(c)}{x - c} > 0. \tag{4}$$

Also, since f has a relative minimum at c, then there exists a neighborhood $V_{\delta_2}(c)$ of c such that if $x \in (V_{\delta_2}(c) \cap I)$, then

$$f(x) \geq f(c). \tag{5}$$

And finally, since c is an interior point of I, there exists a neighborhood $V_{\delta_3}(c)$ of c such that $V_{\delta_3}(c) \subseteq I$. Now let $\delta = \min\{\delta_1, \delta_2, \delta_3\}$, and choose x such that $c - \delta < x < c$. Then $x \in V_{\delta_3}(c)$, so $x \in I$, and also $x \in V_{\delta_1}(c)$ and $x \in V_{\delta_2}(c)$, so both the inequalities (4) and (5) displayed above hold. But since $x < c$, then

$$x - c < 0. \tag{6}$$

This gives a contradiction, because multiplying inequality (4) by inequality (6) gives $f(x) - f(c) < 0$, which contradicts inequality (5).

We have now shown that the assumption $f'(c) > 0$ leads inevitably to a contradiction, whether f has a relative minimum or relative maximum at c.

A similar proof shows that the assumption $f'(c) < 0$ leads to a contradiction. (The difference from the case when $f'(c) > 0$ is that the inequalities (1) and (4) get reversed, so we have to choose x such that $c - \delta < x < c$ to show that f cannot have a relative maximum at c and we have to choose x such that $c < x < c + \delta$ to show that f cannot have a relative minimum at c.)

Since $f'(c)$ cannot be positive and cannot be negative, we must have $f'(c) = 0$. This completes the proof.

Remark. This proof could be shortened a bit by using a technique we've used a couple of times before: first observe that f has a relative maximum at c if and only if the function $-f$ has a relative minimum at c, and that $f'(c) = 0$ if and only if $-f'(c) = 0$. Therefore, once we prove the theorem in the case when f has a relative maximum at c, we can then easily obtain the theorem in the case when f has a relative minimum at c by applying the result about relative maximums to the function $-f$.