Review for Test 3

In reviewing for this test, I would first look over the problems that were assigned and try to re-do the ones that I got wrong, or that didn't get graded and I was unsure about. Then I would check to see whether I could easily do similar problems from the exercise sections, without referring to any kind of solutions manual or using the answers in the back of the book. This is especially important if you got help when you did any of the homework assignments; there's a big difference between doing a problem on your own, and listening to someone else explain how to do it! At the same time I would be reading (hopefully, *re*-reading) each section carefully, stopping to think about anything said in the text that I didn't feel I understood fully. There are some parts of the text that you can skip over because they weren't covered in class; I'll identify these below.

13.1 Three-dimensional coordinate systems. Despite the title of the section, which mentions coordinate *systems*, there is only one three-dimensional coordinate system discussed here: namely, the rectangular coordinate system. (Two other three-dimensional coordinate systems are discussed in section 13.7, which is not covered on Test 3.) To define the rectangular coordinate system you first define the coordinate axes and coordinate planes (see figure 3 on p. 829), and then define the rectangular coordinates x, y, and z of a point P to be the distances from P to the coordinate planes. Thus, x is the distance from P to the yz-plane, y is the distance from P to the xz-plane, and z is the distance from P to the xy-plane. You need to have a clear understanding of the geometrical definition of x, y, and z to be able to understand the rest of chapter 13.

This section also explains where the formula for the distance between two points comes from, and explains why a sphere with radius r and center at (h, k, l) has the equation $(x-h)^2 + (y-k)^2 + (z-l)^2 = r^2$. It's at least as important to understand *why* this is the equation for the sphere as it is to memorize the equation.

Re-read this entire section.

13.2 Vectors. A vector is, as far as we are concerned, an arrow which starts at one point and ends at another point. (Vectors actually have a more sophisticated interpretation which you learn about in physics or engineering courses.) The two points can be in the xy-plane, in which case the vector is two-dimensional, or in xyz-space, in which case the vector is three-dimensional.

Two vectors are considered to be the same if they both point in the same direction and have the same length. Thus the vector starting at (-1,0,6) and ending at (0,2,11) is the same as the vector starting at (3,1,4) and ending at (4,3,9), even though the two vectors start and end at different points. We can describe a vector by giving its components: the vector in the preceding sentence would be expressed as $\mathbf{v} = \langle 1, 2, 5 \rangle$ because to move from its beginning point to its endpoint you travel 1 unit in the x-direction, 2 units in the y-direction, and 5 units in the z-direction.

Notice the difference between the notation $(1, 2, 5)$ and the notation $\langle 1, 2, 5 \rangle$. The notation $(1, 2, 5)$ describes a *point* P located 1 unit away from the yz-plane, 2 units away from the xz-plane, and 5 units above the xy-plane. The notation $\langle 1, 2, 5 \rangle$ describes a *vector* which could start at any point A, and ends at another point B which is 1 unit away from A

in the x-direction, 2 units away from A in the x-direction, and 5 units away from A in the z direction. In particular, $\langle 1, 2, 5\rangle$ is the vector which starts at the origin and ends at the point $(1, 2, 5)$. For this reason, $\langle 1, 2, 5\rangle$ is called the position vector of the point $(1, 2, 5)$. In later sections of this chapter it is important to understand the difference between the position vector of a point, and the point itself. The position vector of a point $P(x, y, z)$ is often written as $\mathbf{r}$, where $\mathbf{r}$ stands for the vector $\langle x, y, z\rangle$.

This section discusses the following three operations: (i) finding the length of a vector, (ii) adding two vectors, and (iii) multiplying a vector by a number. (In this chapter, a number is sometimes called a "scalar". There's no good reason for doing that here. In more advanced courses, numbers, vectors and more complicated objects called "tensors" are all considered as parts of a single framework and it's useful to call numbers "scalars" to call attention to the ways in which they differ from vectors and tensors within this framework.) For each of these three operations, you should know both the geometric interpretation and the formula for computing the result of the operation in terms of the components of the vectors. For example, the geometric interpretation of the addition of two vectors $\mathbf{a}$ and $\mathbf{b}$ is that $\mathbf{a} + \mathbf{b}$ is the vector which starts at the beginning point of $\mathbf{a}$ and ends at the ending point of $\mathbf{b}$, when the ending point of $\mathbf{a}$ is attached to the beginning point of $\mathbf{b}$. See figures 3 and 4 on p. 835. The formula for computing the sum of two vectors in terms of their components is given in the box at the bottom of p. 837.

Read this entire section, except for the paragraph on p. 840 titled "Applications".

13.3 The dot product. In this section, we define another operation between vectors which, like the ones defined in section 13.2 has both a geometric interpretation and a formula for computing the result of the operation in terms of components. The operation is called the *dot product* (not a very descriptive name). The formula for computing the dot product is given in Definition 1 on p. 843, and the geometric interpretation is given in Theorem 3 on p. 844, which says that the dot product of two vectors is equal to the product of the lengths of the vectors and the cosine of the angle between the vectors. In class, I proved Theorem 3. This is a very important and beautiful proof, because it assigns to an algebraic formula (the one in Definition 1) a geometrical interpretation which is not at all obvious when one looks at the formula. However, I won't ask you to reproduce this proof on the test.

One of the main uses of the dot product is to check whether two vectors are orthogonal ("orthogonal" means the same as "perpendicular"). See the box on p. 845.

You should read from the beginning of this section through p. 845. The paragraphs titled "Direction angles and direction cosines" and "Projections" are not covered in this class, so you don't need to read them for the test. (However, the material in these two sections is crucial in understanding applications of vectors to physics and engineering.)

13.4 The cross product. The cross product is yet another operation between vectors which comes with both a formula (Definition 1 on p. 850) and a geometrical interpretation (very last paragraph on p. 852). Again, one has to explain why the formula given in Definition 1 has that geometrical interpretation. The proof in the text is not very enlightening; I tried to give a better proof in class. In any case, you won't need to know either of these proofs for the test.

The main use we have for the cross product in this class is as a way to find a vector which is orthogonal to two given vectors. That is, if you want to find a vector which is orthogonal to both **a** and **b**, then you can just compute $\mathbf{a} \times \mathbf{b}$.

Please avoid the mistake of confusing the cross product with the dot product. It helps to remember that the result of the dot product is a number, and the result of the cross product is a vector. For this reason, the dot product is sometimes called the "scalar product" and the cross product is sometimes called the "vector product". Yet another source of confusion is the fact that you can multiply a vector by a number; this operation is neither a dot product nor a cross product, and is sometimes called "scalar multiplication", not to be confused with "scalar product". All this inelegant terminology is used because of historical reasons: it's hard to change names once they have been entrenched in people's minds.

You should read from the beginning of the section through p. 854. Again, you won't need to know the proofs of Theorems 5, 6, or 8 for the test; although it certainly doesn't hurt to read through them. Also, you don't need to memorize the formulas in Theorem 8, except that you should remember part 1 of Theorem 8: that is, $\mathbf{a} \times \mathbf{b}$ is always equal to the *negative* of $\mathbf{b} \times \mathbf{a}$.

13.5 Equations of lines and planes. In space, a plane is defined by *one* equation and a line is defined by *two* equations. This is because a line can be thought of as the intersection of two planes, so the equations for a line can just be taken to be the pair of equations which define the two planes.

The form for an equation for a plane is easy to remember; if $\mathbf{n} = \langle a, b, c \rangle$ is a vector which is perpendicular to the plane (called a normal vector to the plane) and $P(x_0, y_0, z_0)$ is a point which is on the plane, then an equation for the plane is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

The same equation in vector form is

$$\mathbf{n} \cdot (\mathbf{r} - \mathbf{r}_0)$$

where $\mathbf{r} = \langle x, y, z \rangle$ is the position vector of any point on the plane and $\mathbf{r}_0$ is the position vector of the point P. The *reason* that the equation of the plane has this form is illustrated simply in Figure 6 on p. 861.

The equation for a plane will look different if you use different points on the plane in place of P, or different normal vectors in place of **n**, but it will amount to the same equation after you simplify it. Any equation for the plane can be simplified to the form $ax + by + cz = d$.

Equations for a line can be given in symmetric form, parametric form, or vector form. The vector form and the parametric form are really the same thing: the parametric form is just obtained by writing out the components of the vector form. The symmetric form is obtained by eliminating t from the parametric form. In writing down each of these forms, the first step is to find a vector parallel to the line and a point on the line.

Again, the equations for the a given line can take different forms, depending on what vector you choose which is parallel to the line, and what point you use on the line.

Read the entire section; except you can skip Examples 8, 9, and 10.