Review for Final Exam – Chapter 12

12.1 Sequences. There is a lot of material in this section, but for the purposes of this course, all we need to know from this section is the following:

1) A *sequence* is just the name for a list of numbers, like $\{1, 3, 5, 7, 9, \ldots\}$ which may or may not follow an obvious pattern. The nth term in a sequence is denoted by a_n. Here the "a" is just a letter of the alphabet used to distinguish this sequence from another sequence, which might be denoted by b_n or u_n, and the subscript n is a number which refers to the position of the term in the sequence. For example, suppose we use a_n to denote the nth term in the sequence $\{1, 3, 5, 7, 9, \ldots\}$. Then $a_4 = 7$, because 7 is the 4th term in the sequence. In general, for this sequence, a_n would be given by the formula $a_n = 2n - 1$. One thing to watch out for is that sometimes instead of starting with a_1 as the first term, a sequence can start with a "0th term", written as a_0.
2) The *limit* of a sequence, if it exists, is a number to which a_n approaches arbitrarily closely as n gets larger. A sequence need not have a limit: for example, a_n might grow without bound, as would be the case if $a_n = n^2$, or a_n might stay bounded but just never stay close to any particular number, as would be the case for the sequence $1, -1, 1, -1, \ldots$. (There are more precise definitions of the notion of limit, but the one given above is enough for our purposes.)
3) If a sequence is steadily increasing, so that $a_1 \leq a_2$, $a_2 \leq a_3$, and so on, and the terms in the sequence never get larger than a fixed number M, so that $a_n \leq M$ for every n, then a_n must necessarily approach a limit. (The limit will also be less than or equal to M, of course.)

12.2 Series. If you add up the terms of a sequence one by one, you get another sequence. For example, if you start with the sequence $\{1, 3, 5, 7, 9, \ldots\}$ and add up the terms one by one, you get the sequence $\{1, 1{+}3, 1{+}3{+}5, 1{+}3{+}5{+}7, \ldots\}$, or $\{1, 4, 9, 16, \ldots\}$. The latter sequence is called the sequence of *partial sums* of the original sequence. In general, if we start with a sequence $\{a_1, a_2, a_3, \ldots\}$ and add up the terms one by one, we get the sequence of partial sums $\{a_1, a_1 + a_2, a_1 + a_2 + a_3, \ldots\}$. We usually use the notation s_n to stand for the nth term of the sequence of partial sums. For example, $s_4 = a_1 + a_2 + a_3 + a_4$.

The word "series" is just another name for a sequence of partial sums. When talking about a series, however, we usually refer to its terms as being the terms of the sequence whose numbers were added one by one to obtain the partial sums, rather than the terms of the sequence of the partial sums themselves. Thus for the sequence $\{1, 3, 5, 7, \ldots\}$, given by the formula $a_n = 2n - 1$, the sequence of partial sums $\{1, 4, 9, 16, \ldots\}$ is called the series whose terms are $a_n = 2n - 1$, and is denoted by $\displaystyle\sum_{n=1}^{\infty} a_n$ or $\displaystyle\sum_{n=1}^{\infty}(2n-1)$ for short.

This is confusing, because what we call the "terms" of the series $\sum_{n=1}^{\infty} a_n$ are the same as the terms of the sequence $\{a_1, a_2, a_3, \ldots\}$. The series, however, is not the same as the sequence. In fact the series $\sum_{n=1}^{\infty} a_n$ is actually the same as the sequence $\{s_1, s_2, s_3, \ldots\}$!

There is one other source of confusion. In this chapter we are mainly concerned with the *sums* of series. The sum of a series is defined to be the limit (if it exists) of the sequence of partial sums which constitutes the series. The source of confusion is that traditionally we use the same symbol, $\sum_{n=1}^{\infty} a_n$, to stand for both the sum of the series (which is a number) and the series itself, which is a sequence. Thus it is correct, but very confusing, to say that "the series $\sum_{n=1}^{\infty} a_n$ is a sequence which converges to the number $\sum_{n=1}^{\infty} a_n$"!

In the first part of Chapter 12 we are mainly concerned with the question: given a series $\sum_{n=1}^{\infty} a_n$, where we know a formula for the terms a_n, how can we decide whether the sum of the series exists? If the sum exists, we say the series *converges*, otherwise we say the series *diverges*.

Two important series are introduced in this section: the harmonic series $\sum_{n=1}^{\infty}$ and the geometric series $\sum_{n=1}^{\infty} ar^{n-1}$. In the latter, a and r can stand for any constants. The harmonic series diverges, and the geometric series will converge if $|r| < 1$ and diverge if $|r| \geq 1$. In the case when $|r| < 1$, there is an important formula for the sum of a geometric series:

$$a + ar + ar^2 + ar^3 + \cdots = \frac{a}{1-r}.$$

Also, this section contains the important "Test for Divergence" (see the boxes on p. 754).

You should read the entire section, except that you can skip Example 6 if you like.

12.3 The Integral Test and estimates of sums.

The Integral Test is a simple and useful test for determining whether a series converges. Notice that it only applies to series whose terms are positive and steadily decrease as n increases.

The Integral Test can be used to show that the p-series

$$\sum_{n=1} \frac{1}{n^p}$$

converges when $p \geq 1$. Notice that the p-series is a very specific series; there are lots of series that look kind of like the p-series but are not the p-series.

Read from the beginning of the section through Example 4. (You might also benefit by reading the proof of the Integral Test, an informal version of which you can find in your notes from class, and of formal version of which is at the end of this section.) You can skip the part of the section titled "Estimating the sum of a series".

12.4 The Comparison Tests. This section contains two more tests for deciding whether a series converges. The first is the Comparison Test, in the box on p. 767; the second is the Limit Comparison Test, in the box on p. 768. Both of them only apply to series with positive terms.

Read from the beginning of the section through Example 4. As the examples show, the Limit Comparison Test is more powerful than the Comparison Test. To prove that a series converges using the Comparison Test, you must find another series which converges and has *larger* terms. To prove that a series converges using the Limit Comparison Test, you need only find another series that converges and has terms that are *about the same size.*

Besides reading the Limit Comparison Test, take a look also at exercises number 40 and 41 on p. 771. These show how to use the Limit Comparison Test in cases when the limit involved turns out to be zero or infinity. This is useful, for example, in doing problem 3 on Exam 2.

12.5 Alternating series. An alternating series is one in which the terms alternate between positive and negative. You can use the Alternating Series Test (box on p. 772) to prove that an alternating series converges, if you show that it satisfies two conditions. (It is possible, however, for an alternating series to converge even if it does not satisfy those two conditions; so you can't use the Alternating Series test to prove that a series diverges.)

Read from the beginning of the section through Example 3.

12.6 Absolute convergence and the ratio and root tests. A series converges absolutely if it not only converges, but still converges when all its negative terms are replaced by positive terms of the same size. A series converges conditionally if it is convergent, but fails to converge when all its negative terms are replaced by positive terms of the same size.

The Ratio and Root Tests are useful because they give a good way of testing whether power series converge (see sections 12.8 to 12.10 below). Notice that these tests are not restricted to series with positive terms. However, in applying the Ratio or Root Test to a series with negative terms you must first take the absolute values of the terms, so whenever you use the Ratio Test to prove that a series converges you are actually proving that the series converges absolutely. (If you try to use the Ratio Test on a series which converges conditionally, you will get an inconclusive result.)

Read this entire section. Actually, you do not need to read the last paragraph on "rearrangements" for the exam, but it is worth checking out for its entertainment value: it explains the surprising fact that if you add up the terms in a conditionally convergent series, the final sum you get depends on the order in which you add the terms. That is, you can find a series which converges to 1, and then merely by rearranging the order of the terms, you can get a series which converges to 2!

12.7 Strategy for testing series. You can skip this section, except that you might possibly use the exercises at the end as practice problems.

12.8 Power series. A power series is a special kind of series in which the terms contain not just numbers, but functions of a variable x. They are called power series because the functions they contain are powers of x: x, x^2, x^3, and so on. (There are other important series which contain different functions of x: for example, Fourier series, which are widely used in science and engineering, are series which contain the functions $\sin x$, $\cos x$, $\sin 2x$, $\cos 2x$, $\sin 3x$, $\cos 3x$, and so on.)

The main question we considered for power series was: for which values of the variable x does a given power series converge, and for which values of x does the series diverge? To answer this question completely, you have to do two things. First, apply the Ratio Test to the series: this will tell you that the series converges whenever x is inside a certain interval (a, b) (i.e., when $a < x < b$) and diverges whenever x is outside $[a, b]$ (i.e., when $x < a$ or $x > b$). Second, use one of the other convergence tests (the Ratio Test will not work!) to decide whether the series converges when $x = a$ or $x = b$.

When you get done with this, you have found all the values of x for which the series converges: this set is called the interval of convergence. Thus the interval of convergence is either (a, b) or $(a, b]$ or $[a, b)$ or $[a, b]$, depending on which of the two endpoints $x = a$ and $x = b$ it contains.

Read this whole section.

12.9 Representations of functions as power series. In this section, a few functions are expressed as power series by using the formula for the sum of a geometric series (see above, with a replaced by 1 and r replaced by x):

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots.$$

Read Examples 1, 2, 3, 6, and 7.

12.10 Taylor and Maclaurin series.

Read from the beginning of the section through Example 1. The formulas in the boxes on pp. 797 and 798 should be memorized. Then you can skip to p. 801 and read examples 4, 5, and 6. It would also help to memorize the Maclaurin series for e^x, $\sin x$, and $\cos x$ (see the box at the bottom of p. 803).

12.11 The binomial series. 12.12 Applications of Taylor polynomials. You can skip these two sections.