Review for Test

I. COUNTING

The number of ways to arrange n distinguishable objects in a row is $n!$.

The number of ways to seat n people at a round table is $(n-1)!$.

The number of ways to order the letters in the word MISSISSIPPI is $11!/(4!2!2!)$. Problems of this type are called Mississippi problems, and frequently appear as parts of a larger problem.

The number of ways to arrange exactly r of n distinguishable objects in a row is $n!/(n-r)!$. This is also the number of ways to choose r objects from n distinguishable objects, where the order of choice matters.

The number of ways to choose r objects from n distinguishable objects (where order doesn't matter) is $n!/r!(n-r)!$, also called $\binom{n}{r}$.

The number of ways to choose r objects from n distinguishable objects, where the order of choice doesn't matter and repetition is allowed, is $\binom{n+r-1}{r}$. This number is also called $\langle {n \atop r} \rangle$. This is also the number of ways to put r indistinguishable balls into n distinguishable boxes, or the number of non-negative integer solutions to the equation $x_1 + x_2 + \cdots + x_n = r$.

The number of ways to choose r objects from n distinguishable objects, where the order of choice does matter and repetition is allowed, is n^r. This is also the number of ways to put r distinguishable balls into n distinguishable boxes.

We do not yet have formulas for the number of ways to put indistinguishable balls into indistinguishable boxes, or distinguishable balls into indistinguishable boxes.

The "Principle of Inclusion and Exclusion" (PIE) says that the number of elements in one or more of the finite sets $A, B, C, \ldots Z$, is

$$\begin{aligned}
&+ (|A| + |B| + |C| + \cdots + |Z|) \\
&- (|A \cap B) + |A \cap C| + \cdots + |Y \cap Z|) \\
&+ (|A \cap B \cap C| + |A \cap B \cap C| + \cdots + |X \cap Y \cap Z|) \\
&- (|A \cap B \cap C \cap D) + \cdots + |W \cap X \cap Y \cap Z|) \\
&\vdots \\
&\pm |A \cap B \cap C \cap \cdots \cap Y \cap Z|.
\end{aligned}$$

You can use PIE to find the number of derangements of the numbers 1 to n (arrangements of the numbers 1 to n so that each number is not in its "natural" position). The answer is

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!$$

II. RECURRENCE RELATIONS AND INDUCTION

In class we saw recurrence relations in the contexts of the following problems: the Tower of Hanoi problem; the Josephus problem; the problem of finding the maximum number of regions a plane can be cut into by n lines, or the maximum number of regions space can be cut into by n planes. We also gave a method for solving recurrence relations of the form

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \cdots + c_p a_{n-p}$$

with given initial conditions for $a_0, a_1, \ldots, a_{p-1}$. The idea is to find the roots r_i of the characteristic equation

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \cdots + c_p r^{n-p}$$

and take combinations of $(r_i)^n$ which satisfy the given initial conditions.

We often guessed (or were given) proposed solutions to these recurrence relations, and then had to use mathematical induction to prove that the guesses were indeed solutions.