Review of Trigonometry

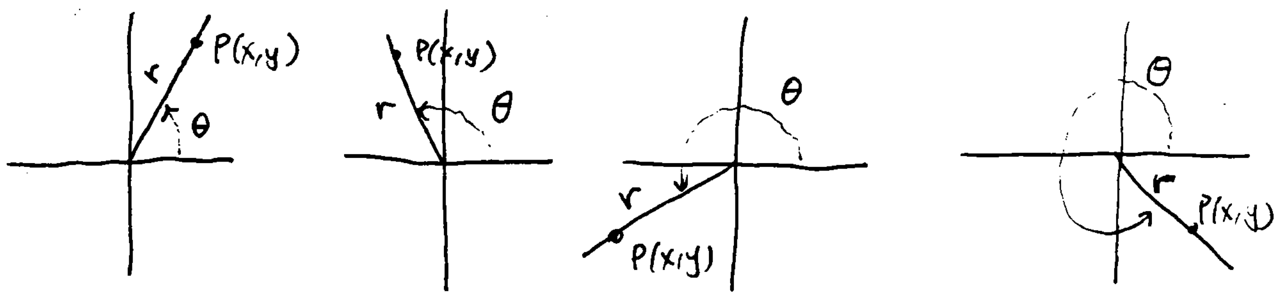
(adapted from "Differential and Integral Calculus" by Ross R. Middlemiss)

1. The definitions. If θ is any number, the trigonometric functions of θ are defined as follows: Construct an angle of θ radians with vertex at the origin and initial side along the positive x-axis, measuring the angle counterclockwise if θ is positive. Choose any point $P(x, y)$ on the terminal side; denote its distance from the origin by r. Then (see figure)

$$\sin\theta = \frac{y}{r}, \qquad \csc\theta = \frac{r}{y},$$

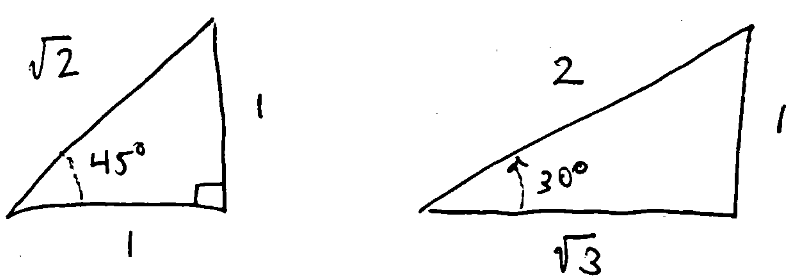
$$\cos\theta = \frac{x}{r}, \qquad \sec\theta = \frac{r}{x},$$

$$\tan\theta = \frac{y}{x}, \qquad \cot\theta = \frac{x}{y}.$$

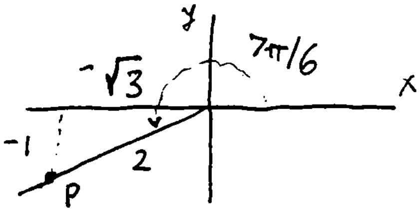


2. The signs of the functions. In the above definitions r is always positive, but x and y can be positive or negative, depending on which quadrant the point $P(x, y)$ is in. Keeping this in mind, we can easily determine whether a given function of a given angle is positive or negative. Thus if α is an angle whose terminal side lies in the second quadrant, $\cos\alpha$ is negative; for $\cos\alpha = x/r$ and x is negative while r is positive.

3. Values of the functions for certain angles. In order to find the value of any function of any angle which is a multiple of 30^o or 45^o, one has only to remember that in the 45^o right triangle the two legs are equal and in the $30^o - 60^o$ right triangle the shortest leg is exactly equal to one half of the hypotenuse. The values of x, y, and r may then be taken as shown in the figure below.



Thus, to find $\sin 210^o$ or, as we prefer, $\sin 7\pi/6$, one draws the figure

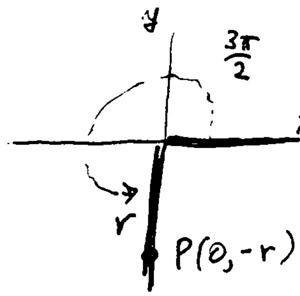


and writes down from it

$$\sin \frac{7\pi}{6} = -\frac{1}{2}.$$

The values of the functions for the quadrantal angles 0, $\pi/2$, π, and $3\pi/2$ can also be easily found. For this, one needs only to note that for such angles either x or y is zero and the other equals $\pm r$. Thus for the angle $3\pi/2$, we have $x = 0$ and $y = -r$; hence (see figure below)

$$\sin \frac{3\pi}{2} = \frac{-r}{r} = -1$$
$$\cos \frac{3\pi}{2} = \frac{0}{2} = 0.$$



4. The values of the functions when that of one is known. If, for a certain angle, the value of one of the six trigonometric functions and the quadrant in which the terminal side lies are known, the values of the other five functions can be found easily. For example, we can easily answer the following questions:

Given that $\sin\theta = -\frac{2}{3}$ and that θ is an angle between 270^o and 360^o, find the value of $15\cos\theta + 6\cos\theta$. (Answer: $2\sqrt{5}$)

If $\tan\theta = 3$ and θ is an angle between 180^o and 270^o, what is the value of $\frac{1}{2}\sec^2\theta - 5\cos 2\theta$? (Answer: 9. For this question we need to know the double angle formula for the cosine, see below.)

5. The fundamental identities. You should memorize and be able to prove the following fundamental relations that exist between the trigonometric functions of θ:

(a) $\sin^2\theta + \cos^2\theta = 1$.

(b) $1 + \tan^2\theta = \sec^2\theta$.

(c) $1 + \cot^2\theta = \csc^2\theta$.

(d) $\tan\theta = \dfrac{\sin\theta}{\cos\theta}$

(e) $\csc\theta = \dfrac{1}{\sin\theta}$, $\sec\theta = \dfrac{1}{\cos\theta}$, $\cot\theta = \dfrac{1}{\tan\theta}$. The proofs follow immediately from the definitions of the functions.

6. The functions of $-\theta$. By comparing the values of x, y, and r for any positive angle θ with those for the negative angle $-\theta$ (an angle equal to θ but measured *clockwise* from the positive x-axis), it is easy to show that

$$\sin(-\theta) = -\sin\theta;$$
$$\cos(-\theta) = \cos\theta;$$
$$\tan(-\theta) = -\tan\theta.$$

The same relations hold if θ is negative and $-\theta$ is positive.

7. The Law of Sines. The Law of Cosines. Addition and subtraction formulas. Double- and half-angle formulas. These are given on page 2 of the "Reference Pages" next to the inside front cover of the textbook. You do not need to memorize these formulas, but you should understand their use.

8. General behavior and graphs of the functions. The functions $\sin x$ and $\cos x$ are continuous for all values of x. The function $\tan x$ is continuous at all values of x except when x is an odd multiple of $\pi/2$; at these points $\tan x$ is undefined.

The manner in which each of these functions varies with x can be discussed in a general way from the corresponding graph. (See "Graphs of Trigonometric Functions" on page 2 of the Reference Pages at the front of the text; these graphs are not very detailed, however.) From the graph of $\sin x$ we see that $\sin x$ increases from 0 at $x = 0$ to 1 at $x = \pi/2$, the *rate* of increase becoming continuously smaller as we move from left to right. In the interval from $x = \frac{1}{2}\pi$ to $x = \frac{3}{2}\pi$, $\sin x$ decreases from $+1$ to -1; it then increases to 0 at $x = -2\pi$. The function is periodic with period 2π; i.e., for any value of x,

$$\sin(x + 2\pi) = \sin x.$$

A similar discussion could be given for the other trigonometric functions based on their graphs.