Maximizers for Strichartz Inequalities on the Torus

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Abstract

We study the existence of maximizers for a one-parameter family of Strichartz inequalities on the torus. In general, maximizing sequences can fail to be precompact in $L^2(\mathbb{T})$, and maximizers can fail to exist. We provide a sufficient condition for precompactness of maximizing sequences (after translation in Fourier space), and verify the existence of maximizers for a range of values of the parameter. Maximizers for the Strichartz inequalities correspond to stable, periodic (in space and time) solutions of a model equation for optical pulses in a dispersion-managed fiber.

1 Introduction

In 1977, in the course of solving a problem on the restrictions of the Fourier transforms of functions on $\mathbb{R}^n$ to quadratic surfaces in $\mathbb{R}^n$, Strichartz [44] obtained an estimate on solutions of the linear Schrödinger equation $v_t - i\Delta_x v = 0$ on $\mathbb{R}^n$, taking the form of an inequality

$$\|v(x,t)\|_{L^q(\mathbb{R}^{n+1})} \leq C\|v(x,0)\|_{L^2(\mathbb{R}^n)}, \tag{1.1}$$

in which $q = 2(n+2)/n$, and the constant C is independent of v. Inequalities such as (1.1) had appeared previously in the literature: for example, [44] references the work of Segal [38], in which an analogous estimate is obtained for the Klein-Gordon equation in $\mathbb{R}^1$; and the periodic version of (1.1) that appears below in (1.2) was already proved by Zygmund in [50]. However, perhaps because Strichartz gave a unified treatment of a family of such estimates, today any inequality which provides a bound on a space-time norm of the solution of a linear dispersive equation is generally termed a Strichartz inequality. For an overview of Strichartz inequalities and their use in the study of partial differential equations, the reader may consult [46] and the references therein.

In this paper we consider a one-parameter family of Strichartz inequalities on the one-dimensional torus $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. The inequalities in question state that for each $B > 0$ there exists a constant $C > 0$ such that for all $u \in L^2(\mathbb{T})$,

$$\left(\int_0^B \int_{\mathbb{T}} |T_t u(x)|^4 \, dx \, dt\right)^{1/4} \leq C \left(\int_{\mathbb{T}} |u(x)|^2 \, dx\right)^{1/2}. \tag{1.2}$$

Here T_t denotes the unitary semigroup defined on $L^2(\mathbb{T})$ by the linear Schrödinger equation. That is, for each function $u \in L^2([0,2\pi])$, $T_t u(x)$ is defined to equal $v(x,t)$, where v is the solution of the linear Schrödinger equation $v_t - iv_{xx} = 0$ on $[0,2\pi]$ with periodic boundary conditions and with initial condition $v(x,0) = u(x)$. As mentioned above, (1.2) was proved by Zygmund in [50] for $B = 2\pi$; the result for general B follows immediately from the result for $B = 2\pi$ via Hölder's inequality and the fact that T_t is periodic in t with period 2π. One can also find Zygmund's original argument reproduced within the proof of Lemma 3.3 below.

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As a general rule, Strichartz inequalities are more elusive in the periodic setting than on the line, due to the fact that in the periodic setting dispersion does not induce decay. Thus, for example, while taking $n = 1$ in (1.1) yields the inequality $\|v(x,t)\|_{L^6(\mathbb{R}^2)} \leq C\|v(x,0)\|_{L^2(\mathbb{R})}$ for solutions of the linear Schrödinger equation on the line, the inequality $\|v(x,t)\|_{L^6(\mathbb{T}^2)} \leq C\|v(x,0)\|_{L^2(\mathbb{T})}$ does not hold for solutions of the linear Schrödinger equation on the torus, for any constant C which is independent of v. On the other hand, a result of Bourgain [6], with important implications for the well-posedness theory of the periodic nonlinear Schrödinger equation, is that one can replace C in the latter inequality by CR^ϵ, where $\epsilon > 0$ is arbitrary and C is independent of v, if one assumes that the Fourier transform of $\hat{v}(k,0)$ of $v(x,0)$ is supported in the ball $\{k \in \mathbb{Z} : |k| \leq R\}$. Bourgain's work has spawned an extensive and rapidly developing theory of Strichartz inequalities on multidimensional tori (see, for example, [36] for a recent survey of some of its aspects). In this paper, however, we confine our attention to the estimate (1.2) on $\mathbb{T}^1$.

Specifically, we are interested here in the question of whether there exists a function $u \in L^2(\mathbb{R})$ for which the best constant in inequality (1.2) is attained. For given $B > 0$, define

$$C_B = \inf\{C > 0 : \text{ inequality (1.2) holds for all } u \in L^2(\mathbb{T})\}. \tag{1.3}$$

If $u \in L^2(\mathbb{T})$ is such that equality holds in (1.2) with $C = C_B$; that is,

$$\left(\int_0^B \int_{\mathbb{T}} |T_t u(x)|^4 \, dx \, dt\right)^{1/4} = C_B \left(\int_{\mathbb{T}} |u(x)|^2 \, dx\right)^{1/2}, \tag{1.4}$$

then we say that u is a *maximizer* for (1.2). This terminology arises from the fact that u maximizes the quantity on the left side of (1.4), subject to the restriction that the L^2 norm of u be held constant. By a *maximizing sequence* for (1.2), we mean a sequence of functions $\{u_j\}$ in $L^2(\mathbb{T})$ such that for some $\lambda > 0$, $\int_{\mathbb{T}} |u_j|^2 \, dx = \lambda$ for all $j \in \mathbb{N}$, while

$$\lim_{j\to\infty} \left(\int_0^B \int_{\mathbb{T}} |T_t u_j(x)|^4 \, dx \, dt\right)^{1/4} = C_B \lambda^{1/2}. \tag{1.5}$$

Depending on the value of B, it is quite possible that maximizers for (1.2) do not exist: in fact, it is not hard to see (cf. Corollary 5.3 below) that if B is of the form $B = N\pi$ with $N \in \mathbb{N}$, then there are no maximizers for (1.2) in $L^2(\mathbb{T})$. On the other hand, if any maximizing sequence happens to converge strongly in $L^2(\mathbb{T})$, then its limit must necessarily be a maximizer, since the left-hand side of (1.5) is a continuous functional on $L^2(\mathbb{T})$.

In general, maximizing sequences do not converge: if there exists more than one maximizer for (1.2), then a maximizing sequence could simply alternate between two maximizers. One might ask whether maximizing sequences are *precompact*, meaning that each of their subsequences has a strongly convergent subsubsequence. This turns out to be false in general, because of the invariance of the left side of (1.2) under the operation of replacing $u(x)$ by $e^{i\theta x}u(x)$, for arbitrary $\theta \in \mathbb{R}$, corresponding to a translation of u in Fourier space. Thus if $u(x)$ is any maximizer, then the sequence $\{e^{ijx}u(x)\}_{j\in\mathbb{N}}$ is a maximizing sequence, and is not precompact.

However, it is not too much to ask that general maximizing sequences be precompact up to translations in Fourier space. Here, we prove as our main result (Theorem 2.1 below) that the inequality $C_B > B/\pi$ is a necessary and sufficient condition for the precompactness, up to translations in Fourier space, of every maximizing sequence for (1.2). As a consequence we obtain a condition for the existence of maximizers which is almost necessary and sufficient (see Corollary 5.1 below), and which we use to obtain that maximizers for (1.2) do exist at least for B in the range $0 < B < B_4$, where $B_4 \approx 2.60$. It remains open whether there is some B in $(0, \pi)$ for which maximizers do not exist.

The existence of maximizers for an analogue of (1.2) on the line,

$$\left(\int_0^1 \int_{\mathbb{R}} |S_t u(x)|^4 \, dx \, dt\right)^{1/4} \leq C \left(\int_{\mathbb{R}} |u(x)|^2 \, dx\right)^{1/2}, \tag{1.6}$$

where S_t denotes the solution operator for the linear Schrödinger equation on $L^2(\mathbb{R})$, has been proved by Kunze in [32]. In fact, Kunze has also in [31] proved the existence of maximizers for Strichartz' original inequality (1.1) in the case $n = 1$, and that existence of maximizers for (1.1) for general n is proved in [39]. Moreover, it has even been proved in [16] and independently in [28] that at least for $n = 1$ and $n = 2$, maximizers for (1.1) are necessarily Gaussians (for other interesting treatments of this result, see [5, 7, 29]).

As explained in [32], an important mathematical feature of the problem of finding maximizers for (1.6) is that maximizing sequences are not, in general, precompact; with the loss of compactness due not just to the invariance of the left side of (1.6) with respect to translations in physical space, but also to its invariance with respect to translations in Fourier space. This necessitated in [32] an elaboration on the method of concentration compactness as used, for example, in Cazenave and Lions' original paper [11]: the classical method of [11] fails to apply directly to maximizing sequences of (1.6), because even when they are tight in physical space, such sequences can still fail to be tight in Fourier space. In fact, in [32], Kunze succeeds not only in proving the existence of maximizers for (1.6), but also in characterizing the way in which maximizing sequences can lose compactness. By virtue of the results of [32] one sees that a sequence of functions $\{u_j\}$ is a maximizing sequence for (1.6) if and only if every subsequence $\{u_{j_m}\}$ has a subsubsequence $\{u_{j_k}\}$ such that, for some sequences $\{\theta_k\}$ and $\{x_k\}$, the sequence $\{e^{i\theta_k} u_{j_k}(x - x_k)\}$ converges in $L^2(\mathbb{R})$ to a maximizing function.

In the periodic case, there is a similar difficulty due to loss of compactness of maximizing sequences $\{u_j\}$ for (1.2). We show below (see Theorem 2.1) that if $C_B > B/\pi$, then every maximizing sequence $\{u_j\}$ for (1.2) must have a subsequence $\{u_{j_k}\}$ such that $\{e^{i\theta_k x} u_{j_k}(x)\}$ converges strongly in $L^2(\mathbb{T})$, for suitably chosen $\{\theta_k\}$. Our proof follows the framework of that given for the nonperiodic case in [32]: first, concentration-compactness arguments are used to show that maximizing sequences must have subsequences which, after translation, are simultaneously tight in physical space and in Fourier space, after which a decomposition of the translated subsequence into high- and low-frequency parts is used to deduce strong convergence in $L^2(\mathbb{R})$. However, the application of this technique to the problem on the torus runs into a difficulty which is not encountered for the problem on the line: in the periodic case, for certain values of B (including $B = 2\pi$) maximizing sequences can vanish or exhibit splitting in Fourier space, while this cannot happen for maximizing sequences on the line. That the difficulty is essential, and not just an artifact of the method of proof, is shown by the fact that, as mentioned above, maximizers do not exist in the periodic case for certain values of B. This seems to be an instance of the general principle that the effects of dispersion in wave propagation are more subtle and delicate in the periodic case than on the line.

We note that although the approach used in [32] to rule out possible loss of compactness in maximizing sequences is sufficient for our purposes, alternative approaches are available, such as those used in [15, 19, 20, 26], which may lead to a shorter proof of our main result.

The validity of the condition $C_B > B/\pi$ can be verified by finding an appropriate test function: it suffices to find $w \in L^2(\mathbb{T})$ such that the ratio of $\|T_t w(x)\|_{L^4(\mathbb{T}\times[0,B])}$ to $\|w\|_{L^2(\mathbb{T})}$ is greater than B/π. Below we obtain existence results by finding suitable test functions which satisfy this condition, in the more convenient form given in Corollary 5.1.

Besides the work mentioned above on maximizers for (1.1) and (1.6), there has been intensive study recently on extremizers for other Strichartz inequalities, often motivated by the relation between Strichartz inequalities and Fourier restriction inequalities. As noted in Strichartz' original paper [44], the classical inequality (1.1) is equivalent to the statement that

$$\|\mathcal{F}_{n+1}(f\sigma)\|_{L^q(\mathbb{R}^{n+1})} \leq C\|f\|_{L^2(\mathcal{M},\sigma)} \tag{1.7}$$

for all $f \in L^2(\mathcal{M})$, where $q = 2(n+2)/n$, $\mathcal{M}$ is the paraboloid in $\mathbb{R}^{n+1}$ given by $\mathcal{M} = \{(t,x) : t = |x|^2, x \in \mathbb{R}^n\}$, σ is the pullback to $\mathcal{M}$ via the projection $(t,x) \mapsto x$ of Lebesgue measure on $\mathbb{R}^n$, and $\mathcal{F}_{n+1}$ is the $(n+1)$-dimensional Fourier transform. By duality, the "Fourier extension inequality" (1.7) is in turn equivalent to the Fourier restriction inequality

$$\|\mathcal{F}_{n+1}f\|_{L^2(\mathcal{M},\sigma)} \leq C\|f\|_{L^p(\mathbb{R}^{n+1})}, \tag{1.8}$$

valid for $f \in L^p(\mathbb{R}^{n+1})$ with $p = 2(n+2)/(n+4)$, the conjugate exponent to q. Hence the results referenced above on maximizers of (1.1) can also be viewed as results on the maximizers of (1.7) or (1.8).

The problem of obtaining inequalities such as (1.7) and (1.8) for other submanifolds $\mathcal{M}$ of $\mathbb{R}^{n+1}$, and in other function spaces on $\mathbb{R}^{n+1}$ or on $\mathcal{M}$, has long been a mainstream topic in harmonic analysis; see for example the survey in [45]. More recently, much attention has been paid to the study of extremizers: for a recent review of some of this work, including an account of its relation to other topics in analysis, see [18]. We mention here, by way of illustration, some results for the case when $\mathcal{M}$ is S^n, the unit sphere in $\mathbb{R}^{n+1}$, and σ is surface measure on S^n. In this case, inequality (1.7) is equivalent to the classical Stein-Tomas inequality [42, 47], valid for $q \geq q_n = 2(n+2)/(n+4)$. The existence of maximizing functions was proved for $q > q_n$, for all $n \in \mathbb{N}$, in [15]. In the much more difficult endpoint case, $q = q_n$, existence of maximizers and precompactness of maxiizing sequences up to symmetries was proved in [13] for $n = 2$, and in [40] and [19] for the case $n = 1$. The question of existence of maximizers is still open for $n \geq 3$, but Frank, Lieb, and Sabin in [19] give an interesting necessary and sufficient condition for existence of maximizers and precompactness of maximizing sequences up to modulations: i.e., up to multiplication of functions $f(\omega)$ on S^n by functions of the form $e^{ix\cdot\omega}$, where $x \in \mathbb{R}^{n+1}$. Their condition takes the form of an inequality relating the best constant in the Stein-Tomas inequality (1.7) to the best constant in the Strichartz inequality (1.1). In [19] it is conjectured that the condition is actually satisfied for every $n \in \mathbb{N}$, and it is shown that this conjecture follows from another conjecture (made by Foschi in [16] and proved there for $n = 2$ and $n = 3$), stating that maximizers for (1.1) on $\mathbb{R}^n$ are given by certain Gaussians.

Concerning the uniqueness of maximizers for (1.7) when $\mathcal{M} = S^n$, Foschi in [17] has proved that when $q = 4$ and $n = 2$, all maximizers are given, up to modulation, by constant functions; thus settling a question raised by Christ and Shao in [13]. Similar uniqueness results are obtained in [8] for $n \in \{3, 4, 5, 6\}$ when $q = 4$, and in [37] for $n \in \{2, 3, 4, 5, 6\}$ when q is an even integer and $q \geq 6$.

As just three examples of the many other recent papers dealing with other choices of $\mathcal{M}$, we mention the work of Stovall [43], Carneiro et al. [9], and Frank and Sabin [20], for the cases in which $\mathcal{M}$ is a paraboloid, a hyperboloid, and a cubic curve, respectively.

The periodic Strichartz inequality (1.2) may also be interpreted as a Fourier restriction inequality. Let $\mathbb{T}^2 = [0, 2\pi] \times [0, 2\pi]$, and let $\mathcal{F}_{t,x}$ denote the joint Fourier transform in the variables (t, x) on $\mathbb{T}^2$, defined below in (2.1). A duality argument shows that the assertion that (1.2) holds for all $u \in L^2(\mathbb{T})$, with best constant C_B defined in (1.3), is equivalent to the assertion that

$$\left(\sum_{n \in \mathbb{Z}} \left| \mathcal{F}_{t,x} g[-n^2, n] \right|^2 \right)^{1/2} \leq D \|g\|_{L^{4/3}(\mathbb{T}^2)}$$

holds for all $g \in L^2(\mathbb{T}^2)$ such that $g(t, x) = 0$ whenever $B \leq t \leq 2\pi$, with best constant D_B given by $D_B = (2\pi)^{3/2} C_B$. In other words, (1.2) amounts to an inequality on restrictions of Fourier transforms on $\mathbb{T}^2$ to a certain parabolic subset $\mathcal{N}$ of the lattice $\mathbb{Z} \times \mathbb{Z}$. Because of the discrete structure of $\mathcal{N}$, the recent literature on extremizers of Fourier restriction inequalities on $\mathbb{R}^n$, in which the key issue to be dealt with is the subtle way in which the geometry of $\mathcal{M}$ is related to possible loss of compactness of maximizing sequences, does not seem to be directly relevant to the problem considered here. However, certain interesting analogies may still be made. For example, the condition for existence of extremizers given in [19] plays a similar role there to the role played here by the condition $C_B > B/\pi$; in that both conditions rule out certain ways in which maximizing sequences can lose compactness. (See the introduction of [19] for a discussion of the role of such "energy" inequalities in other problems in the calculus of variations.)

An additional motivation for studying maximizers of Strichartz inequalities is that they often represent important solutions of partial differential equations arising in mathematical physics. In the case of inequality (1.2), maximizers correspond to ground-state solutions of a equation, sometimes known as the dispersion-managed nonlinear Schrödinger equation (DMNLS), which models nonlinear, long-wavelength light pulses in a dispersion-managed optical fiber. In the non-periodic case, where pulses are defined on the entire real line and decay as $|x| \to \infty$ (note that x is actually a time variable in this model), the DMNLS equation was derived in [21] (see also [2]), and the existence of ground-state solutions was proved by variational methods in [49] for the case of positive average dispersion, and in [32] for the case of zero average dispersion. For more results on the existence and properties of ground-state solutions of the DMNLS and related equations on the line, see [12, 14, 23, 24, 25, 26, 27, 33, 41].

The maximizers whose existence is proved in the present paper, by contrast, correspond to solutions of an equation which models pulses in a dispersion-managed fiber which are periodic in both x and t. This periodic DMNLS equation was derived in [3], where well-posedness results for the initial-value problem are proved for the case of positive and zero average dispersion, and results on the existence and stability of periodic ground-state solutions were proved in both cases.

In the case of zero average dispersion, the periodic DMNLS equation, for complex-valued functions $u(x,t)$ which are periodic with period L in the x variable, can be written in Hamiltonian form as

$$u_t = -i\nabla H_L(u). \tag{1.9}$$

Here the Hamiltonian functional $H_L : L^2(\mathbb{T}) \to \mathbb{R}$ is given by

$$H_L(u) = -\frac{2\pi}{L}\int_0^L \int_0^1 |T_t^L u(x)|^4 \ dt \ dx,$$

and ∇H_L denotes the gradient of H_L, given by

$$\nabla H_L(u) = -\frac{8\pi}{L}\int_0^L \int_0^1 T_{-t}^L \left(|T_t^L u(x)|^2 T_t^L u(x)\right) \ dt \ dx.$$

The operator T_t^L appearing in the integrand is the solution operator for the linear Schrödinger equation with periodic boundary conditions on $0 \le x \le L$. That is, $T_t^L(u)(x) = v(x,t)$, where $v(x,t)$ is periodic with period L in x and satisfies the equation $iv_t + v_{xx} = 0$, with initial condition $v(x,0) = u(x)$. (The gradient here is defined with respect to the real-valued inner product $\langle u, v\rangle$ defined on $L^2(\mathbb{T})$ by

$$\langle u, v\rangle = \Re \int_{\mathbb{T}} u(x)\overline{v}(x) \ dx.$$

That is, we have

$$\lim_{\epsilon\to 0} \frac{H_L(u+\epsilon v) - H_L(u)}{\epsilon} = \langle \nabla H_L(u), v\rangle$$

for all $v \in L^2(\mathbb{T})$.)

As an immediate consequence of our results on existence of maximizers for (1.2), we obtain results on the existence and stability of sets of ground-state solutions to the periodic DMNLS equation (1.9), for a range of values of the period L. Ground-state solutions can be characterized as solutions of the form $u(x,t) = e^{i\omega t}\phi(x)$, where $\omega \in \mathbb{R}$ and the profile function $\phi(x)$ minimizes $H_L(u)$ among all functions in $L^2(0,L)$ with fixed L^2 norm λ. For each fixed value of $\lambda > 0$, the stability of the set $S_{L,\lambda}$ of corresponding ground-state profile functions follows from a standard argument, once we have shown that every minimizing sequence for the associated variational problem converges strongly to $S_{L,\lambda}$ in L^2 norm.

The organization of the remainder of this paper is as follows. In Section 2, we establish notation and state our main results. Section 3 contains some preliminary lemmas. The proof of Theorem 2.1, on the sufficiency of the condition $C_B > B/\pi$ for the existence of maximizers, is given in Section 4. This sufficient condition is verified for a range of values of B in Section 5. The final Section 6 discusses the implications for existence and stability of non-empty sets of ground-state solutions of the periodic DMNLS equation (1.9).

2 Notation and Main Results

If E is a measurable subset of $\mathbb{R}$ and $1 \le p < \infty$, we define $L^p(E)$ to be the space of Lebesgue measurable complex-valued functions u on E such that $\|u\|_{L^p(E)} = \left(\int_E |u|^p \ dx\right)^{1/p}$ is finite. We denote by $L^2(\mathbb{T})$ the space of Lebesgue measurable, square-integrable, 2π-periodic functions on $\mathbb{R}$.

We can identify $L^2(\mathbb{T})$ with $L^2([0,2\pi])$. We will often denote the norm of u in $L^2(\mathbb{T})$ simply by $\|u\|_{L^2}$. For $B>0$, we define $L^p_{t,x}([0,B]\times\mathbb{T})$ to be the space of all functions $f(t,x)$ defined for $(t,x)\in[0,B]\times\mathbb{T}$ such that the norm $\|f\|_{L^p_{t,x}([0,B]\times\mathbb{T})}=\left(\int_0^B\int_{\mathbb{T}}|f(t,x)|^p\ dx\ dt\right)^{1/p}$ is finite.

For $1\le p<\infty$, we define $\ell^p(\mathbb{Z})$ to be the space of sequences of complex numbers $\{a(n)\}_{n\in\mathbb{Z}}$ such that $\|a\|_{\ell^p}=\left(\sum_{n\in\mathbb{Z}}|a(n)|^p\right)^{1/p}$ is finite. We define $\ell^\infty(\mathbb{Z})$ to be the space of all sequences $\{a(n)\}_{n\in\mathbb{Z}}$ such that $\|a\|_{\ell^\infty}=\sup_{n\in\mathbb{Z}}|a(n)|$ is finite.

For $u\in L^2(\mathbb{T})$, we define the Fourier transform of u to be the sequence $\mathcal{F}u$ in $l^2(\mathbb{Z})$ given by

$$\mathcal{F}u[n]=\frac{1}{2\pi}\int_{\mathbb{T}}e^{-inx}u(x)\ dx$$

for $n\in\mathbb{Z}$. We also denote $\mathcal{F}u[n]$ by $\hat{f}(n)$. The inversion formula for the Fourier transform is given by

$$u(x)=\sum_{n\in\mathbb{Z}}\hat{u}(n)\ e^{inx}.$$

The correspondence $u\to\hat{u}$ defines a one-to-one map from $L^2(\mathbb{T})$ onto $\ell^2(\mathbb{Z})$; and with this definition of the Fourier transform, Parseval's theorem asserts that for any $u,v\in L^2(\mathbb{T})$, one has

$$\int_{\mathbb{T}}u(x)\bar{v}(x)\ dx=2\pi\sum_{n\in\mathbb{Z}}\hat{u}(n)\bar{\hat{v}}(n),$$

and in particular

$$\|u\|_{L^2}=\sqrt{2\pi}\|\hat{u}\|_{\ell^2}.$$

Also, the Fourier transform of the product uv is given by a convolution:

$$\mathcal{F}(uv)[n]=(\hat{u}*\hat{v})[n]=\sum_{k\in\mathbb{Z}}\hat{u}(k)\hat{v}(n-k).$$

In Section 6, we will have occasion to mention the action of the Fourier transform on functions of period L. Define $L^2_{\text{per}}(0,L)$ to be the set of all measurable functions on $\mathbb{R}$ which are periodic of period L and which are square integrable on $0\le x\le L$. For $u\in L^2_{\text{per}}(0,L)$, we define the Fourier transform $\mathcal{F}_Lu\in\ell^2(\mathbb{Z})$ by

$$\mathcal{F}_Lu[n]=\frac{1}{L}\int_0^L e^{-i(2\pi n/L)x}u(x)\ dx,$$

and we have the have the Fourier inversion formula

$$u(x)=\sum_{n\in\mathbb{Z}}\mathcal{F}_Lu[n]\ e^{i(2\pi n/L)x}.$$

We define the Sobolev space $H^1=H^1(\mathbb{T})$ to be the space of all functions $u\in L^2(\mathbb{T})$ such that the H^1 norm

$$\|u\|_{H^1}=\left(\sum_{n\in\mathbb{Z}}|n|^2|\hat{u}(n)|^2\right)^{1/2}$$

is finite.

We denote by $\mathcal{D}$ the set of all functions $u\in L^2(\mathbb{T})$ such that $\hat{u}(n)=0$ for all but finitely many $n\in\mathbb{Z}$. In particular, functions in $\mathcal{D}$ are infinitely smooth.

For functions $g(t,x)\in L^2_{t,x}([0,2\pi]\times\mathbb{T})$, the space-time Fourier transform of g is the sequence $\mathcal{F}_{t,x}g\in\mathbb{Z}\times\mathbb{Z}$ defined by

$$\mathcal{F}_{t,x}g[m,n]=\frac{1}{(2\pi)^2}\int_0^{2\pi}\int_{\mathbb{T}}e^{-imt}e^{-inx}g(t,x)\ dt\ dx. \tag{2.1}$$

The correspondence $g \to \mathcal{F}_{t,x}g$ defines a one-to-one map from $L^2([0,2\pi]\times\mathbb{T})$ onto the space $\ell^2(\mathbb{Z}\times\mathbb{Z})$ of square-integrable sequences $b[m,n]$, and Parseval's theorem asserts that for $g_1, g_2 \in L^2_{t,x}([0,2\pi]\times\mathbb{T})$, one has

$$\int_{\mathbb{T}}\int_0^{2\pi} g_1\overline{g_2}\ dt\ dx = (2\pi)^2\sum_{m\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}\mathcal{F}_{t,x}g_1\overline{\mathcal{F}_{t,x}g_2}.$$

For $t\in\mathbb{R}$, define $T_t : L^2(\mathbb{T})\to L^2(\mathbb{T})$ as a Fourier multiplier operator by setting, for $u\in L^2(\mathbb{T})$ and $n\in\mathbb{Z}$,

$$\mathcal{F}(T_t u)[n] = e^{-in^2 t}\mathcal{F}u[n]. \tag{2.2}$$

For a given $u\in L^2(\mathbb{T})$, $T_t u(x)$ is thus defined as a measurable function of x and t. Since $\int_{\mathbb{T}}|T_t u(x)|^2\ dx = \|\mathcal{F}(T_t u)\|_{\ell^2}^2 = \|\mathcal{F}u[n]\|_{\ell^2}^2 = \|u\|_{L^2(\mathbb{T})}^2$ for each $t\in\mathbb{R}$, we have $\int_0^B\int_{\mathbb{T}}|T_t u(x)|^2\ dt\ dx < \infty$ for every $B>0$. Therefore $T_t u\in L^2_{t,x}([0,B]\times\mathbb{T})$. In particular, taking $B=2\pi$, we have that $\mathcal{F}_{t,x}(T_t u)$ is well-defined in $\ell^2(\mathbb{Z}\times\mathbb{Z})$ and is given by

$$\mathcal{F}_{t,x}(T_t u)[m,n] = \begin{cases}\hat{u}(n) & \text{if } m=-n^2\\ 0 & \text{if } m\neq -n^2.\end{cases}$$

Fix $B>0$, and for $u\in L^2(\mathbb{T})$, define

$$W_B(u) = \int_0^B\int_{\mathbb{T}}|T_t u(x)|^4\ dx\ dt. \tag{2.3}$$

We consider the variational problem of maximizing $W_B(u)$ over $L^2(\mathbb{T})$, subject to the constraint $\|u\|_{L^2}^2 = \lambda$, where $\lambda>0$ is fixed. Define

$$J_{B,\lambda} = \sup\ \{W_B(u) : u\in L^2(\mathbb{T}) \text{ and } \|u\|_{L^2}^2 = \lambda\}. \tag{2.4}$$

We say that a sequence $\{u_j\}_{j\in\mathbb{N}}$ in $L^2(\mathbb{T})$ is a maximizing sequence for $J_{B,\lambda}$ if $\|u_j\|_{L^2}^2=\lambda$ for all $j\in\mathbb{N}$ and $\lim_{j\to\infty}W_B(u_j) = J_{B,\lambda}$; and we say that $u_0\in L^2(\mathbb{T})$ is a maximizer for $J_{B,\lambda}$ if $\|u_0\|_{L^2}^2=\lambda$ and $W_B(u_0)=J_{B,\lambda}$.

Observe that since $W_B(\lambda u) = \lambda^4 W_B(u)$ for all $\lambda\in\mathbb{R}$, it follows that

$$J_{B,\lambda} = \lambda^2 J_{B,1} \tag{2.5}$$

for all $\lambda>0$. In other words, if we define $C_B = J_{B,1}$, then for all $u\in L^2(\mathbb{T})$ we have

$$W_B(u) \le C_B\|u\|_{L^2}^4, \tag{2.6}$$

which is equivalent to the Strichartz inequality (1.2) with best constant C_B. That C_B is indeed finite is shown below in Lemma 3.3. It is clear that the existence of a maximizing function for $C_B = J_{B,1}$ is equivalent to the existence of a maximizing function for $J_{B,\lambda}$ for every $\lambda>0$.

The following theorem establishes a sharp condition for the precompactness, up to translations in Fourier space, of maximizing sequences for $J_{B,1}$.

Theorem 2.1.
(i) For all $B>0$,

$$J_{B,1} \ge \frac{B}{\pi}.$$

(ii) If $B>0$ and

$$J_{B,1} > \frac{B}{\pi}, \tag{2.7}$$

then every maximizing sequence for $J_{B,1}$ has a subsequence which, after translations in Fourier space, converges in $L^2(\mathbb{T})$ to a maximizer for $J_{B,1}$. That is, if $\{u_j\}$ is a sequence such that $\|u_j\|_{L^2}=1$ for *all $j \in \mathbb{N}$ and $\lim_{j\to\infty} W_B(u_j) = J_{B,1}$, then there exists a subsequence $\{u_{j_k}\}$ and a sequence of real numbers $\{\theta_k\}$ such that $\{e^{i\theta_k x} u_{j_k}(x)\}$ converges strongly in $L^2(\mathbb{T})$.*

In particular, there does exist a maximizer for $J_{B,1}$: that is, there exists $u_0 \in L^2(\mathbb{T})$ such that $\|u_0\|_{L^2} = 1$ and $W_B(u_0) = J_{B,1}$.

(iii) If $B > 0$ and

$$J_{B,1} = \frac{B}{\pi}, \tag{2.8}$$

then there exist maximizing sequences for $J_{B,1}$ which do not have any subsequences that can be made to converge by translating the terms in Fourier space.

As corollaries of this result, we obtain existence and non-existence results for maximizers of $J_{B,1}$ for certain values of B. In Section 5 below we show that $J_{B,1} > B/\pi$ is true at least for all B in some range $0 < B < B_4$, where $B_4 \approx 2.6$; and therefore $J_{B,1}$ does have maximizers for B in this range (see Corollary 5.4). On the other hand, we see that $J_{B,1} = B/\pi$ for all B of the form $B = N\pi$, where $N \in \mathbb{N}$, and hence for these values of B, no maximizer for $J_{B,1}$ exists (see Corollary 5.3).

Remark. Although the questions of whether maximizers exist for $J_{B,1}$, and whether $J_{B,1}$ is equal to B/π, are somewhat subtle; it is easy to answer the corresponding questions for *minimizers* of $W_B(u)$ subject to the constraint that $\|u\|_{L^2} = 1$. In fact, for every $B > 0$ the minimum is equal to $B/2\pi$, and is attained at the constant function $v(x) \equiv 1/(2\pi)$ on $\mathbb{T}$. To see this, note that by Hölder's inequality, if $\|u\|_{L^2} = 1$ then

$$B = \int_0^B \int_{\mathbb{T}} |u|^2 \, dx \, dt = \int_0^B \int_{\mathbb{T}} |T_t u|^2 \, dx \, dt \le \sqrt{W_B(u)}\sqrt{2\pi B},$$

which implies that $W_B(u) \ge B/(2\pi) = W_B(v)$.

By a well-known argument, the assertions of Theorem 2.1 yield results on the existence and stability of sets of ground-state solutions of the periodic DMNLS equation (1.9), for a range of values of the period L. We review these arguments below in Section 6, where we show (see Theorem 6.2) that for all $L \in (0, 2\pi/\sqrt{B_4})$, equation (1.9) has a one-parameter family $\{S_{L,\lambda} : \lambda > 0\}$ of non-empty sets $S_{L,\lambda}$ of ground-state profiles, and each set $S_{L,\lambda}$ is stable with respect to the flow defined by (1.9).

On the other hand, the nonexistence of maximizers of $J_{B,1}$ when B is an integer multiple of π translates into a nonexistence result for ground-state solutions of (1.9): when L is of the form $L = 2\sqrt{\pi/N}$ for some $N \in \mathbb{N}$, then (1.9) can have no ground-state solutions (see Theorem 6.4).

3 Preliminary results

An important property of W_B is that it is invariant with respect to translations in Fourier space as well as translations in physical space.

Lemma 3.1. *Let $B > 0$.*

(i) Suppose $u \in L^2(\mathbb{T})$ and $x_0 \in \mathbb{T}$. If we define $v \in L^2(\mathbb{T})$ by $v(x) = u(x - x_0)$ for $x \in \mathbb{T}$, then for all $(t, x) \in \mathbb{R} \times \mathbb{T}$, we have $T_t v(x) = T_t u(x - x_0)$. In particular,

$$W_B(v) = W_B(u).$$

(ii) Suppose $u \in L^2(\mathbb{T})$ and $n_0 \in \mathbb{Z}$. If we define $w \in L^2(\mathbb{T})$ by setting $\hat{w}(n) = \hat{u}(n - n_0)$ for all $n \in \mathbb{Z}$, then for all $(t, x) \in \mathbb{R} \times \mathbb{T}$, we have

$$T_t w(x) = e^{in_0 x} e^{-in_0^2 t} T_t u(x - 2n_0 t).$$

In particular

$$W_B(w) = W_B(u).$$

(iii) If $\{u_j\}_{j\in\mathbb{N}}$ is a maximizing sequence for $J_{B,1}$ in $L^2(\mathbb{T})$, $\{m_j\}_{j\in\mathbb{N}}$ is a sequence of integers, and $\{x_j\}_{j\in\mathbb{N}}$ is a sequence in $\mathbb{T}$, then $\{e^{im_jx}u_j(x-x_j)\}_{j\in\mathbb{N}}$ is also a maximizing sequence for $J_{B,1}$. Also, if u is a maximizer for $J_{B,1}$, then $e^{imx}u(x-x_0)$ is also a maximizer, for every $m\in\mathbb{Z}$ and every $x\in\mathbb{T}$.

Proof. The statements in (i) and (ii) follow easily from the definition of T_t as a Fourier multiplier operator. We note that the invariance of W_B under translations in Fourier space also follows immediately from the formula given below for W_B in (3.3).

Part (iii) of the Lemma follows immediately from parts (i) and (ii), since the norm in $L^2(\mathbb{T})$ is also invariant under translations in both physical space and Fourier space. □

We now state a version of Lions' concentration compactness lemma.

Lemma 3.2. *Fix $M>0$, and suppose that for each $j\in\mathbb{N}$, $\{a_j(n)\}_{n\in\mathbb{Z}}$ is an element of $\ell^2(\mathbb{Z})$ such that $\|a_j\|_{\ell^2}^2=M$. Then the sequence $\{a_j\}_{j\in\mathbb{N}}$ in $\ell^2(\mathbb{Z})$ has a subsequence, still denoted by $\{a_j\}$, for which exactly one of the following three alternatives holds:*

1. *(Vanishing) For every $r\in\mathbb{N}$,*

$$\lim_{j\to\infty}\sup_{m\in\mathbb{Z}}\sum_{n=m-r}^{m+r}|a_j(n)|^2=0.$$

2. *(Splitting) There is an $\alpha\in(0,M)$ with the following property: for every $\delta>0$, there exist numbers $r_1,r_2\in\mathbb{N}$ with $r_2-r_1\ge\delta^{-1}$, sequences $\{b_j\}_{j\in\mathbb{N}}$ and $\{c_j\}_{j\in\mathbb{N}}$ in $\ell^2(\mathbb{Z})$, and an integer sequence $\{m_j\}_{j\in\mathbb{N}}$ such that for all $j\in\mathbb{N}$,*

$$b_j(n)=0 \text{ for all } n\in\mathbb{Z} \text{ such that } |n-m_j|>r_1,$$

$$c_j(n)=0 \text{ for all } n\in\mathbb{Z} \text{ such that } |n-m_j|<r_2,$$

$$\|a_j-(b_j+c_j)\|_{\ell^2}^2\le\delta,$$

$$\big|\|b_j\|_{\ell^2}^2-\alpha\big|\le\delta,$$

and

$$\big|\|c_j\|_{\ell^2}^2-(M-\alpha)\big|\le\delta.$$

3. *(Tightness) There exist integers $\{m_j\}_{j\in\mathbb{N}}$ such that for every $\epsilon>0$, there exists $r\in\mathbb{N}$ so that*

$$\sum_{n=m_j-r}^{m_j+r}|a_j(n)|^2\,dx\ge M-\epsilon$$

for all $j\in\mathbb{N}$.

We omit the proof of Lemma (3.2), which is standard: for example, except for obvious modifications it is the same as the proof given for Lemma 3.1 of [32]. However, for future reference we emphasize here that the three alternatives given in Lemma 3.1 are mutually exclusive. In particular, if there exist integers $\{m_j\}$ such that the translated sequence $\{\tilde a_j\}=\{a_j(\cdot-m_j)\}$ converges strongly in $\ell^2(\mathbb{Z})$, then all subsequences of $\{a_j\}$ are tight, and no subsequence of $\{a_j\}$ vanishes. For indeed, if $\{\tilde a_j\}$ converges in ℓ^2 norm to a limit $a\in\ell^2(\mathbb{N})$, then we must have $\|a\|_{\ell^2}^2=M>0$, and therefore for every $\epsilon>0$ there exists $r\in\mathbb{N}$ such that $\sum_{n=-r}^{r}|a(n)|^2>M-\epsilon$. From the strong convergence of $\{\tilde a_j\}$ to a in ℓ^2, it then follows that

$$\sum_{n=m_j-r}^{m_j+r}|a_j(n)|^2>M-\epsilon$$

for all sufficiently large j. This implies that all subsequences of $\{a_j\}$ are tight, and that no subsequence can vanish.

The following lemma gives a Fourier decomposition of $W_B(u)$ which will be important in analyzing the behavior of maximizing sequences. All sums which appear are intended to be performed over all integral values of the index of summation, unless otherwise specified.

Lemma 3.3. *Suppose $B > 0$.*

(i) There exists $C > 0$ such that for all $u \in L^2(\mathbb{T})$,

$$G_B(u) \leq C\|u\|^4_{L^2(\mathbb{T})}, \tag{3.1}$$

where

$$G_B(u) = \sum_l \sum_n \sum_p \frac{|\hat{u}(n)\hat{u}(n-l)\hat{u}(n-p)\hat{u}(n-p-l)|}{1+|lp|B}. \tag{3.2}$$

(ii) For all $u \in L^2(\mathbb{T})$, we have

$$W_B(u) = 2\pi \sum_l \sum_n \sum_p \hat{u}(n)\bar{\hat{u}}(n-l)\bar{\hat{u}}(n-p)\hat{u}(n-p-l)\int_0^B e^{-2ilpt}\, dt, \tag{3.3}$$

where the sum on the right-hand side converges absolutely.

Moreover, there exists $C > 0$ such that for all $u \in L^2(\mathbb{T})$,

$$W_B(u) \leq C\|u\|^4_{L^2(\mathbb{T})}. \tag{3.4}$$

(iii) For all $u \in L^2(\mathbb{T})$, we have

$$W_B(u) = 4\pi B\|\hat{u}\|^4_{\ell^2} - 2\pi B\|\hat{u}\|^4_{\ell^4} + D_B(u), \tag{3.5}$$

where

$$D_B(u) = 2\pi \sum_{l\neq 0} \sum_n \sum_{p\neq 0} \hat{u}(n)\bar{\hat{u}}(n-l)\bar{\hat{u}}(n-p)\hat{u}(n-p-l)\int_0^B e^{-2ilpt}\, dt. \tag{3.6}$$

(The sum on the right-hand side converges absolutely.)

Proof. Suppose $u \in L^2(\mathbb{T})$. We can decompose the triple sum which defines $G_B(u)$ into two parts (I) and (II), where (I) represents the sum taken over all $(l, n, p) \in \mathbb{Z}^3$ for which $|p| < |l|$, and (II) represents the sum taken over all (l, n, p) for which $|p| \geq |l|$.

Define $K : \mathbb{Z} \to \mathbb{R}$ by $K(n) = 1/(1 + |n|^2 B)$, and note that $K \in \ell^1(\mathbb{Z})$. If $|p| < |l|$, then we have $1/(1 + |lp|B) \leq K(p)$. Therefore we can use Holder's inequality and Young's convolution inequality to make the estimate

$$\begin{aligned}
(I) &\leq \sum_n \sum_p K(p)\,|\hat{u}(n)\hat{u}(n-p)| \sum_l |\hat{u}(n-l)\hat{u}(n-p-l)| \\
&\leq \|\hat{u}\|^2_{\ell^2} \sum_n |\hat{u}(n)| \sum_p K(p)\,|\hat{u}(n-p)| \\
&= \|\hat{u}\|^2_{\ell^2} \sum_n |\hat{u}(n)|\,(K * |\hat{u}|)(n) \\
&\leq \|\hat{u}\|^2_{\ell^2}\|\hat{u}\|_{\ell^2}\|K * |\hat{u}|\|_{\ell^2} \\
&\leq \|\hat{u}\|^4_{\ell^2}\|K\|_{\ell^1} \leq C\|u\|^4_{L^2(\mathbb{T})}.
\end{aligned} \tag{3.7}$$

On the other hand, when $|p| \geq |l|$, we have $1/(1 + |lp|B) \leq K(l)$, so we can write

$$(II) \leq \sum_n \sum_l K(l)\,|\hat{u}(n)\hat{u}(n-l)| \sum_p |\hat{u}(n-p)\hat{u}(n-p-l)|,$$

and then use the same argument as for (I), only with l and p interchanged, to show that $(II) \leq C\|u\|^4_{L^2(\mathbb{T})}$. This then proves part (i) of the Lemma.

To prove part (ii), suppose first that $u \in \mathcal{D}$, the space of all functions $u \in L^2(\mathbb{T})$ such that $\hat{u}$ is compactly supported in $\mathbb{Z}$, so that in particular $T_t u$ is bounded on $\mathbb{T}$ for all $t \in \mathbb{R}$, and all the computations which follow are readily justified. Writing $T_t u = \sum_{n\in\mathbb{Z}} \hat{u}(n) e^{i(nx-n^2t)}$, we obtain

$$\begin{aligned} W_B(u) &= \|T_t u\|^4_{L^4_{t,x}([0,B]\times\mathbb{T})} = \|T_t u \cdot \overline{T_t u}\|^2_{L^2_{t,x}([0,B]\times\mathbb{T})} \\ &= \left\| \sum_n \sum_m \hat{u}(n)\bar{\hat{u}}(m) e^{i((n-m)x-(n^2-m^2)t)} \right\|^2_{L^2_{t,x}([0,B]\times\mathbb{T})} \\ &= \left\| \sum_n \sum_l \hat{u}(n)\bar{\hat{u}}(n-l) e^{ilx} e^{-il(2n-l)t} \right\|^2_{L^2_{t,x}([0,B]\times\mathbb{T})}, \end{aligned}$$

where in the last step we used $l = n - m$ as an index of summation. Now letting

$$b(l,t) = \sum_n \hat{u}(n)\bar{\hat{u}}(n-l) e^{-il(2n-l)t},$$

we can write

$$W_B(u) = \int_0^B \int_{\mathbb{T}} \left| \sum_l b(l,t) e^{ilx} \right|^2 \, dx \, dt.$$

Using Parseval's theorem, we obtain that

$$\begin{aligned} W_B(u) &= 2\pi \int_0^B \sum_l |b(l,t)|^2 \, dt \\ &= 2\pi \sum_l \sum_n \sum_r \hat{u}(n)\bar{\hat{u}}(n-l)\hat{u}(r-l)\bar{\hat{u}}(r) \int_0^B e^{-il(2n-2r)t} \, dt. \end{aligned} \tag{3.8}$$

Changing the index of summation in the innermost sum to $p = n-r$ yields the sum on the right-hand side of (3.3). Thus we have proved that (3.3) holds, at least in the case when $u \in \mathcal{D}$.

In light of the fact that

$$\left| \int_0^B e^{-2i\theta t} \, dt \right| \leq \frac{2B}{1+|\theta|B} \tag{3.9}$$

for all $\theta \in \mathbb{R}$ and all $B > 0$, it follows from what we have proved that there exists $C > 0$ such that

$$W_B(u) \leq C G_B(u)$$

and therefore

$$W_B(u) = \|T_t u\|^4_{L^4_{t,x}([0,B]\times\mathbb{T})} \leq C\|u\|^4_{L^2(\mathbb{T})} \tag{3.10}$$

for all $u \in \mathcal{D}$.

Given any $u \in L^2(\mathbb{T})$, define a sequence $\{u_N\} \in \mathcal{D}$ by setting $\widehat{u_N}(n) = \hat{u}(n)$ for $|n| \leq N$ and $\widehat{u_N}(n) = 0$ for $|n| > N$. Then

$$W_B(u_N) = 2\pi \sum_l \sum_n \sum_p \widehat{u_N}(n)\overline{\widehat{u_N}}(n-l)\overline{\widehat{u_N}}(n-p)\widehat{u_N}(n-p-l) \int_0^B e^{-2ilpt} \, dt \tag{3.11}$$

holds for each $N \in \mathbb{N}$.

By Parseval's theorem, $T_t u_N$ converges to $T_t u$ in $L^2_{t,x}([0,2\pi]\times\mathbb{T})$ as $N \to \infty$. It follows from (3.10) that $T_t u_N$ also converges to $T_t u$ in $L^2_{t,x}([0,B]\times\mathbb{T})$ for every $B \in [0,2\pi]$, and then by periodicity for every $B \in \mathbb{R}$. Also, by what we have proved, $\{T_t u_N\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $L^4_{t,x}([0,B]\times\mathbb{T})$, and so converges in the norm of $L^4_{t,x}([0,B]\times\mathbb{T})$ to some limit, which must therefore equal $T_t u$. It follows that

$$W_B(u) = \lim_{N\to\infty} W_B(u_N).$$

On the other hand, it follows from part (i) of the Lemma, (3.9), (3.11), and the Dominated Convergence Theorem that

$$\lim_{N\to\infty} W_B(u_N) = 2\pi \sum_l \sum_n \sum_p \hat{u}(n)\bar{\hat{u}}(n-l)\bar{\hat{u}}(n-p)\hat{u}(n-p-l)\int_0^B e^{-2ilpt}\, dt,$$

with the sum on the right-hand side converging absolutely. Therefore part (ii) of the Lemma has been proved.

To prove part (iii), we proceed by splitting the sum in (3.3) into four parts, according to whether p and l are zero or nonzero.

First, we sum over all values of l, n, and p such that $l \neq 0$ and $p = 0$. This gives

$$\begin{aligned}
2\pi B \sum_{l\neq 0}\sum_n |\hat{u}(n)\hat{u}(n-l)|^2 &= 2\pi B \sum_n \sum_{m\neq n} |\hat{u}(n)\hat{u}(m)|^2 \\
&= 2\pi B \left(\sum_n \sum_m |\hat{u}(n)\hat{u}(m)|^2 - \sum_n |\hat{u}(n)|^4 \right) \\
&= 2\pi B \left(\|\hat{u}\|^4_{\ell^2} - \|\hat{u}\|^4_{\ell^4} \right).
\end{aligned}$$

Second, we sum over all values of l, n, and p such that $l = 0$ and $p \neq 0$, obtaining the same result as above: that is,

$$2\pi B \left(\|\hat{u}\|^4_{\ell^2} - \|\hat{u}\|^4_{\ell^4} \right).$$

Third, we sum over all values of l, n, and r such that $l = 0$ and $p = 0$, resulting in

$$2\pi B \sum_n |\hat{u}(n)|^4 = 2\pi B \|\hat{u}\|^4_{\ell^4}.$$

Finally, if we sum over all values of l, n, and p such that $l \neq 0$ and $p \neq 0$, we obtain the sum in (3.6) which defines $D_B(u)$. (Note that the absolute convergence of this sum is guaranteed by part (ii) of the Lemma.) Taking the sum of all four parts, we obtain the result (3.5), completing the proof of the Lemma. □

For what follows, we note that if $\{u_j\}$ is a sequence in $L^2(\mathbb{T})$ such that $\|u_j\|^2_{L^2} = 1$ for all $j \in \mathbb{N}$, then by Parseval's theorem, we have that $\{\hat{u}_j\}$ is a sequence in $\ell^2(\mathbb{Z})$ with $\|\hat{u}_j\|^2_{\ell^2} = \frac{1}{2\pi}$, and therefore we can apply Lemma 3.2 to $\{\hat{u}_j\}$ with $M = \frac{1}{2\pi}$.

Lemma 3.4. *Let $\{u_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{T})$ be a sequence such that $\|u_j\|^2_{L^2} = 1$ for all $j \in \mathbb{N}$. Suppose that the sequence $\{\widehat{u_j}\}$ in $\ell^2(\mathbb{Z})$ vanishes in the sense of Lemma 3.2. Then $\|\widehat{u}_j\|^4_{\ell^4} \to 0$ as $j \to \infty$.*

Proof. If $\{\widehat{u_j}\}$ vanishes, then for each $r \in \mathbb{N}$ and for each $\epsilon > 0$, there exists $N = N(r,\epsilon) \in \mathbb{N}$ such that

$$\sup_{m\in\mathbb{Z}} \sum_{n=m-r}^{m+r} |\widehat{u_j}(n)|^2 < \frac{\epsilon}{2}$$

for $j \geq N$. In particular, for $j \geq N$ we have that $|\widehat{u_j}(n)|^2 < \dfrac{\epsilon}{2}$ for all $n \in \mathbb{Z}$. This implies that $\|\widehat{u_j}\|_{\ell^\infty} \to 0$ as $j \to \infty$. Since $\|\widehat{u_j}\|^4_{\ell^4} \leq \|\widehat{u_j}\|^2_{\ell^2}\|\widehat{u_j}\|^2_{\ell^\infty}$, it follows that $\|\widehat{u}_j\|^4_{\ell^4} \to 0$ as $j \to \infty$.

□

Definition 3.5. *For $u_1, u_2, u_3, u_4 \in L^2(\mathbb{T})$, define*

$$F(u_1,u_2,u_3,u_4) = \int_0^B \int_{\mathbb{T}} T_t u_1\ \overline{T_t u_2}\ T_t u_3\ \overline{T_t u_4}\ dx\ dt.$$

Lemma 3.6. *There exists $C>0$ such that for all functions u_1, u_2, u_3, and u_4 in $L^2(\mathbb{T})$,*

$$|F(u_1,u_2,u_3,u_4)| \le C\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}\|u_3\|_{L^2_x}\|u_4\|_{L^2_x}.$$

Proof. By Holder's inequality and Lemma 3.3, we have

$$\begin{aligned}\int_0^B\int_{\mathbb{T}}\left|T_tu_1\ \overline{T_tu_2}\ T_tu_3\ \overline{T_tu_4}\right|\ dx\ dt &\le \|T_tu_1\|_{L^4_{s,x}}\ \|T_tu_2\|_{L^4_{t,x}}\ \|T_tu_3\|_{L^4_{t,x}}\ \|T_tu_4\|_{L^4_{t,x}}\\ &\le C\|u_1\|_{L^2_x}\|u_2\|_{L^2_x}\|u_3\|_{L^2_x}\|u_4\|_{L^2_x}.\end{aligned}$$

□

Lemma 3.7. *There exists $C>0$ such that for all $u,v,w,h\in L^2(\mathbb{T})$ with $u=v+w+h$ and $\|h\|_{L^2}\le 1$, we have*

$$\begin{aligned}\left|\|T_tu\|^4_{L^4_{t,x}}-\|T_tv\|^4_{L^4_{t,x}}-\|T_tw\|^4_{L^4_{t,x}}\right| &\le C\left(1+\|u\|^3_{L^2}+\|v\|^3_{L^2}+\|w\|^3_{L^2}\right)\|h\|_{L^2}+\\ &+4F(v,v,w,w)+F(w,v,w,v)+F(v,w,v,w)+\\ &+2\left[F(v,v,v,w)+F(v,v,w,v)+F(v,w,w,w)+F(w,v,w,w)\right].\end{aligned} \tag{3.12}$$

Proof. For $u,v,w,h\in L^2(\mathbb{T})$, we have

$$\|T_tu\|^4_{L^4_{t,x}}-\|T_tv\|^4_{L^4_{t,x}}-\|T_tw\|^4_{L^4_{t,x}}=\int_0^B\int_{\mathbb{T}}|T_tv+T_tw+T_th|^4-|T_tv|^4-|T_tw|^4\ dx\ dt.$$

By writing the integrand on the right hand side of the above equation as

$$[T_tv+T_tw+T_th]^2[\overline{T_tv}+\overline{T_tw}+\overline{T_th}]^2-|T_tv|^4-|T_tw|^4$$

and expanding, we obtain

$$\int_0^B\int_{\mathbb{T}}|T_tv+T_tw+T_th|^4-|T_tv|^4-|T_tw|^4\ dx\ dt=A+B+R,$$

where A is a finite sum of terms of the form $\int_0^B\int_{\mathbb{T}}T_tf_1\ \overline{T_tf_2}\ T_tf_3\ \overline{T_th}\ dx\ dt$ and B is a finite sum of terms of the form $\int_0^B\int_{\mathbb{T}}\overline{T_tf_1}\ T_tf_2\ \overline{T_tf_3}\ T_th\ dx\ dt$, with $f_1,f_2,f_3\in\{u,v,w,h\}$, and R consists of the seven terms involving F on the right side of (3.12). We apply the triangle inequality, Lemma 3.6, and Young's inequality to the terms in A and B to get the desired result. □

Lemma 3.8. *The map $W_B: L^2(\mathbb{T})\to\mathbb{R}$ is continuous.*

Proof. Take $w=0$ in Lemma 3.7. □

Lemma 3.9. *There exists $C>0$ such that for all $v,w\in L^2(\mathbb{T})$, all $\delta>0$, and all integers n_0, r_1, and r_2, if $r_2-r_1\ge\delta^{-1}$, $\hat{v}(n)=0$ for $|n-n_0|>r_1$, and $\hat{w}(n)=0$ for $|n-n_0|<r_2$, then*

$$\begin{aligned}|F(v,v,w,w)| &\le (2\pi B+C\delta^{\frac12})\|\hat{v}\|^2_{\ell^2}\ \|\hat{w}\|^2_{\ell^2},\\ |F(w,v,w,v)| &\le C\|\hat{v}\|^2_{\ell^2}\ \|\hat{w}\|^2_{\ell^2}\ \delta^{\frac12},\\ |F(v,w,v,w)| &\le C\|\hat{v}\|^2_{\ell^2}\ \|\hat{w}\|^2_{\ell^2}\ \delta^{\frac12},\\ |F(v,v,v,w)| &\le C\|\hat{v}\|^3_{\ell^2}\ \|\hat{w}\|_{\ell^2}\ \delta^{\frac12},\\ |F(v,v,w,v)| &\le C\|\hat{v}\|^3_{\ell^2}\ \|\hat{w}\|_{\ell^2}\ \delta^{\frac12},\\ |F(v,w,w,w)| &\le C\|\hat{v}\|_{\ell^2}\ \|\hat{w}\|^3_{\ell^2}\ \delta^{\frac12},\\ |F(w,w,w,v)| &\le C\|\hat{v}\|_{\ell^2}\ \|\hat{w}\|^3_{\ell^2}\ \delta^{\frac12}.\end{aligned} \tag{3.13}$$

Proof. For any $u_1, u_2, u_3, u_4 \in L^2(\mathbb{T})$, we have by Fubini's theorem and Parseval's theorem that

$$\begin{aligned}
F(u_1,u_2,u_3,u_4) &= 2\pi \int_0^B \sum_n \mathcal{F}(T_t u_1 \overline{T_t u_2} T_t u_3)[n]\ \overline{\widehat{u_4}}(n)\ dt \\
&= 2\pi \int_0^B \sum_n e^{in^2 t} \left(\widehat{T_t u_1} * \widehat{\overline{T_t u_2}} * \widehat{T_t u_3}\right)[n]\ \overline{\widehat{u_4}}(n)\ dt \\
&= 2\pi \int_0^B \sum_n \sum_{n_1} \sum_{n_2} e^{in^2 t}\ \widehat{T_t u_1}(n-n_1-n_2)\ \widehat{\overline{T_t u_2}}(n_1)\ \widehat{T_t u_3}(n_2)\ \overline{\widehat{u_4}}(n)\ dt \\
&= 2\pi \sum_{n_1} \sum_{n_2} \sum_{n_3}\ \widehat{u_1}(n_3)\ \widehat{\overline{u_2}}(n_1)\ \widehat{u_3}(n_2)\ \overline{\widehat{u_4}}(n_1+n_2+n_3) \int_0^B e^{-2it(n_1+n_3)(n_1+n_2)}\ dt.
\end{aligned} \tag{3.14}$$

where all of the sums are taken over $\mathbb{Z}$, and in the last expression we used a new index of summation $n_3 = n - n_1 - n_2$. Taking $u_1 = u_2 = v$ and $u_3 = u_4 = w$ in (3.14), we get

$$|F(v,v,w,w)| \le 2\pi \sum_{n_1}\sum_{n_2}\sum_{n_3} \left| \widehat{v}(n_3)\widehat{\overline{v}}(n_1)\widehat{w}(n_2)\overline{\widehat{w}}(n_1+n_2+n_3) \int_0^B e^{-2it(n_1+n_3)(n_1+n_2)}\ dt \right|. \tag{3.15}$$

We write the right-hand side of (3.15) as the sum of four parts, $(I)+(II)+(III)+(IV)$, where (I) is the sum over all terms for which $|n_1+n_3| = 0$; (II) is the sum over the terms for which $|n_1+n_2| = 0 < |n_1+n_3|$; (III) is the sum over the terms for which $|n_1+n_2| \ge |n_1+n_3| \ge 1$; and (IV) is the sum over the terms for which $|n_1+n_3| > |n_1+n_2| \ge 1$.

Then we have

$$(I) = 2\pi B \sum_{n_2}\sum_{n_3} \left| \widehat{v}(n_3)\overline{\widehat{v}}(n_3)\widehat{w}(n_2)\overline{\widehat{w}}(n_2) \right| = 2\pi B \|\widehat{v}\|_{\ell^2}^2\ \|\widehat{w}\|_{\ell^2}^2. \tag{3.16}$$

For (II), we have

$$(II) \le 2\pi B \sum_{n_2}\sum_{n_3} \left| \widehat{v}(n_3)\widehat{\overline{v}}(-n_2)\widehat{w}(n_2)\overline{\widehat{w}}(n_3) \right| = 2\pi B \sum_{n_2}\sum_{n_3} |\widehat{v}(n_3)\widehat{v}(n_2)\widehat{w}(n_2)\widehat{w}(n_3)| = 0, \tag{3.17}$$

because the assumptions on the supports of v and w in Lemma 3.9 imply that $\widehat{v}(n_2)\widehat{w}(n_2) = 0$ for all $n_2 \in \mathbb{Z}$.

Before obtaining estimates for (III) and (IV) we note that, in light of the assumptions on the supports of $\widehat{v}$ and $\widehat{w}$, for $\widehat{v}(n_3)\ \widehat{\overline{v}}(n_1)\ \widehat{w}(n_2)\ \overline{\widehat{w}}(n_1+n_2+n_3)$ to be nonzero we must have $|n_3 - n_0| \le r_1$, $|n_1 + n_0| \le r_1$, $|n_2 - n_0| \ge r_2$, and $|n_1+n_2+n_3-n_0| \ge r_2$.

To estimate (III), we first observe that if $|n_1+n_2| \ge |n_1+n_3| \ge 1$, then in all nonzero terms of the sum,

$$\begin{aligned}
1 + |n_1+n_2|\,|n_1+n_3| &\ge |n_1+n_2|\,|n_1+n_3| \\
&\ge (|n_1+n_2+n_3-n_0| - |n_3-n_0|)^{\frac{1}{2}}\ |n_1+n_3|^{\frac{3}{2}} \\
&\ge (r_2 - r_1)^{\frac{1}{2}}\ |n_1+n_3|^{\frac{3}{2}} \\
&\ge \delta^{-1/2} |n_1+n_3|^{\frac{3}{2}}.
\end{aligned}$$

Define $K_1(n) = \chi_{|n|\ge 1}\ |n|^{-\frac{3}{2}}$, so that $\|K_1\|_{\ell^1} < \infty$, and define $K_2(n) = K_1(n)\,[(|\hat{w}(-.)| * |\overline{\widehat{w}}|)(n)]$.

Using (3.9), we can write

$$
\begin{aligned}
(III) &\le C \sum_{n_1}\sum_{n_2}\sum_{n_3} \left[\frac{\chi_{|n_1+n_2|\ge|n_1+n_3|\ge 1}}{1+|n_1+n_2||n_1+n_3|}\right] \left|\widehat{v}(n_3)\widehat{\overline{v}}(n_1)\widehat{w}(n_2)\widehat{\overline{w}}(n_1+n_2+n_3)\right| \\
&\le C\delta^{\frac{1}{2}} \sum_{n_1}\sum_{n_3} \chi_{|n_1+n_3|\ge 1}\ |n_1+n_3|^{-\frac{3}{2}} \left||\widehat{v}(n_3)\widehat{\overline{v}}(n_1)\right| \sum_{n_2} \left|\widehat{w}(n_2)\widehat{\overline{w}}(n_1+n_2+n_3)\right| \\
&= C\delta^{\frac{1}{2}} \sum_{n_1}\sum_{n_3} K_1(n_1+n_3) \left|\widehat{v}(n_3)\widehat{\overline{v}}(n_1)\right| \left(\left|\widehat{w}(-.)\right| * \left|\widehat{\overline{w}}\right|\right)(n_1+n_3) \\
&= C\delta^{\frac{1}{2}} \sum_{n_1}\sum_{n_3} K_2(n_1+n_3) \left|\widehat{v}(n_3)\right| \left|\widehat{\overline{v}}(n_1)\right| \\
&= C\delta^{\frac{1}{2}} \sum_{n_1} \left|\widehat{\overline{v}}(-n_1)\right| \ (K_2 * \left|\widehat{v}(-.)\right|)(n_1) \\
&\le C\delta^{\frac{1}{2}}\ \|\widehat{v}\|_{\ell^2} \left\|K_2 * \left|\widehat{v}(-.)\right|\right\|_{\ell^2} \\
&\le C\delta^{\frac{1}{2}} \|\widehat{v}\|_{\ell^2}^2 \|K_2\|_{\ell^1} \\
&\le C\delta^{\frac{1}{2}} \|\widehat{v}\|_{\ell^2}^2 \|K_1\|_{\ell^1}\ \|\hat{w}\|_{\ell^2}^2 \\
&\le C\delta^{\frac{1}{2}} \|\hat{v}\|_{\ell^2}^2 \|\hat{w}\|_{\ell^2}^2,
\end{aligned}
\tag{3.18}
$$

where Young's inequality was used in the last few estimates.

To estimate (IV), we observe that if $|n_1+n_3| > |n_1+n_2| \ge 1$, then in all nonzero terms of the sum,

$$
\begin{aligned}
1+|n_1+n_2|\,|n_1+n_3| \ge |n_1+n_2|^2 & \\
&= |(n_1+n_2+n_3-n_0)-(n_3-n_0)|^{\frac{1}{2}}\ |n_1+n_2|^{\frac{3}{2}} \\
&\ge (2\delta^{-1})^{\frac{1}{2}}\ |n_1+n_2|^{\frac{3}{2}}.
\end{aligned}
$$

This time we let $K_3 = K_1(n)(|\hat{v}(-.)| * |\widehat{\overline{w}}|)(n)$ with K_1 as previously defined, and we follow a similar argument as the one used to estimate (III) to obtain

$$
\begin{aligned}
(IV) &\le 2\pi C \sum_{n_1}\sum_{n_2}\sum_{n_3} \left[\frac{\chi_{|n_1+n_3|\ge|n_1+n_2|\ge 1}}{1+|n_1+n_2||n_1+n_3|}\right] \left|\widehat{v}(n_3)\widehat{\overline{v}}(n_1)\widehat{w}(n_2)\widehat{\overline{w}}(n_1+n_2+n_3)\right| \\
&\le C\delta^{\frac{1}{2}} \|\hat{v}\|_{\ell^2}^2 \|\hat{w}\|_{\ell^2}^2.
\end{aligned}
\tag{3.19}
$$

Taking the sum of the estimates in (3.16), (3.17), (3.18) and (3.19) now gives the desired estimate for $F(v,v,w,w)$.

To illustrate the proofs of the remaining estimates in (3.13), consider for example the estimate for $F(v,v,w,v)$. Taking $u_1 = u_2 = u_4 = v$ and $u_3 = w$ in (3.14), we get

$$
\begin{aligned}
|F(v,v,w,v)| &\le 2\pi \sum_{n_1}\sum_{n_2}\sum_{n_3} \left|\widehat{v}(n_3)\widehat{\overline{v}}(n_1)\widehat{w}(n_2)\widehat{\overline{v}}(n_1+n_2+n_3) \int_0^B e^{-2it(n_1+n_3)(n_1+n_2)}\ dt\right| \\
&= (I)+(II)+(III)+(IV),
\end{aligned}
\tag{3.20}
$$

where (I) to (IV) are defined in the same way as in the paragraph following (3.15). The sums (III) and (IV) in (3.20) can be estimated in the same way as the analogous sums in (3.15), and the same argument used to prove (3.17) shows that $(II) = 0$ here as well. In contrast to (3.16), however, here we find that $(I) = 0$. Indeed, we can write

$$
(I) = 2\pi B \sum_{n_2}\sum_{n_3} \left|\widehat{v}(n_3)\widehat{\overline{v}}(n_3)\widehat{w}(n_2)\widehat{\overline{v}}(n_2)\right|.
$$

Because of our assumptions on the supports of $\widehat{v}$ and $\widehat{w}$, we have $\widehat{v}(n_2)\widehat{w}(n_2) = 0$ for all $n_2 \in \mathbb{Z}$; and therefore $(I) = 0$. It follows that the desired estimate holds for $F(v,v,w,v)$.

The proofs of the remaining estimates in (3.13) proceed in the same way as the proof of the estimate for $F(v,v,w,v)$. □

Lemma 3.10. *Let $\{u_j\}_{j\in\mathbb{N}} \subset L^2(\mathbb{T})$ be a sequence such that $\|u_j\|_{L^2}^2 = 1$ for all $j \in \mathbb{N}$. Suppose that the sequence $\{\widehat{u_j}\}$ in $\ell^2(\mathbb{Z})$ vanishes in the sense of Lemma 3.2. Then $D_B(u_j) \to 0$ as $j \to \infty$.*

Proof. Let $\beta \geq 1$. From the proof of Lemma 3.3, we see that there exists C depending only on B such that

$$|D_B(\widehat{u_j})| \leq C \sum_{l\neq 0} \sum_{n} \sum_{p\neq 0} \frac{1}{1+|lp|} |\widehat{u_j}(n)\widehat{u_j}(n-l)\widehat{u_j}(n-p)\widehat{u_j}(n-p-l)|$$

We can decompose the triple sum on the right-hand side into three parts, writing it as $(I)+(II)+(III)$, where (I) is the sum over all (l,n,p) such that $1 \leq |l| \leq \beta$ and $1 \leq |p| \leq \beta$, (II) is the sum over all (l,n,p) such that $|p| > |l| \geq 1$ and $|p| > \beta$, and (III) is the sum over all (l,n,p) such that $|l| > \beta$ and $|l| \geq |p| \geq 1$.

To estimate (I), we write

$$\begin{aligned}
(I) &\leq \sum_n |\widehat{u_j}(n)| \sum_{l=-\beta}^{\beta} |\widehat{u_j}(n-l)| \sum_{p=-\beta}^{\beta} |\widehat{u_j}(n-p)\widehat{u_j}(n-p-l)| \\
&\leq \sum_n |\widehat{u_j}(n)| \sum_{l=-\beta}^{\beta} |\widehat{u_j}(n-l)| \left(\sum_{p=-\beta}^{\beta} |\widehat{u_j}(n-p)|^2\right)^{1/2} \|\widehat{u_j}\|_{\ell^2} \\
&\leq \|\widehat{u_j}\|_{\ell^2} \sup_{m\in\mathbb{Z}} \left(\sum_{r=m-\beta}^{m+\beta} |\widehat{u_j}(r)|^2\right)^{1/2} \sum_n |\widehat{u_j}(n)| \sum_{l=-\beta}^{\beta} |\widehat{u_j}(n-l)| \\
&\leq \|\widehat{u_j}\|_{\ell^2} \sup_{m\in\mathbb{Z}} \left(\sum_{r=m-\beta}^{m+\beta} |\widehat{u_j}(r)|^2\right)^{1/2} \|\widehat{u_j}\|_{\ell^2} \|\chi_{[-\beta,\beta]} * |\widehat{u_j}|\|_{\ell^2} \\
&\leq 2\beta \|\widehat{u_j}\|_{\ell^2}^3 \sup_{m\in\mathbb{Z}} \left(\sum_{r=m-\beta}^{m+\beta} |\widehat{u_j}(r)|^2\right)^{1/2}.
\end{aligned}$$

To estimate (II) we observe that for all (l,n,p) which appear in that sum,

$$1+|lp| > |l|^{\frac{3}{2}}|p|^{\frac{1}{2}} > \beta^{\frac{1}{2}}|l|^{\frac{3}{2}}.$$

We write

$$\begin{aligned}
(II) &= \sum_l \sum_n \sum_p \frac{\chi_{\{|p|>|l|\geq 1\}}(l,p)}{1+|lp|} |\widehat{u_j}(n)\widehat{u_j}(n-l)\widehat{u_j}(n-p)\widehat{u_j}(n-p-l)| \\
&\leq \beta^{-\frac{1}{2}} \sum_n |\widehat{u_j}(n)| \sum_l \chi_{|l|\geq 1}(l)|l|^{-\frac{3}{2}} |\widehat{u_j}(n-l)| \sum_p |\widehat{u_j}(n-p)\widehat{u_j}(n-p-l)| \\
&\leq \beta^{-\frac{1}{2}} \|\widehat{u_j}\|_{\ell^2}^2 \sum_n |\widehat{u_j}(n)| \sum_l \chi_{|l|\geq 1}(l)|l|^{-\frac{3}{2}} |\widehat{u_j}(n-l)|.
\end{aligned}$$

Therefore, if we define $K_1(l) = \chi_{|l|\geq 1}(l)|l|^{-\frac{3}{2}}$ and apply Young's convolution inequality, we obtain that

$$(II) \leq \beta^{-\frac{1}{2}} \|\widehat{u_j}\|_{\ell^2}^3 \|K_1 * |\widehat{u_j}|\|_{\ell^2} \leq \beta^{-\frac{1}{2}} \|\widehat{u_j}\|_{\ell^2}^3 \|K_1\|_{\ell^1} \|\widehat{u_j}\|_{\ell^2} \leq C\beta^{-\frac{1}{2}} \|\widehat{u_j}\|_{\ell^2}^4.$$

In estimating (III) we can use that

$$1+|lp| > |l|^{\frac{1}{2}}|p|^{\frac{3}{2}} > \beta^{\frac{1}{2}}|p|^{\frac{3}{2}}.$$

We write

$$
\begin{aligned}
(III) &= \sum_l \sum_n \sum_p \frac{\chi_{\{|l|\geq|p|\geq 1\}}(l,p)}{1+|lp|} |\widehat{u_j}(n)\widehat{u_j}(n-l)\widehat{u_j}(n-p)\widehat{u_j}(n-p-l)| \\
&\leq \beta^{-\frac{1}{2}} \sum_n |\widehat{u_j}(n)| \sum_p \chi_{\{|p|\geq 1\}}(p)|p|^{-\frac{3}{2}} |\widehat{u_j}(n-p)| \sum_l |\widehat{u_j}(n-l)\widehat{u_j}(n-p-l)| \\
&\leq \beta^{-\frac{1}{2}} \|\widehat{u_j}\|_{\ell^2}^2 \sum_n |\widehat{u_j}(n)| \sum_p \chi_{\{|p|\geq 1\}}(p)|p|^{-\frac{3}{2}} |\widehat{u_j}(n-p)| \\
&= \beta^{-\frac{1}{2}} \|\widehat{u_j}\|_{\ell^2}^2 \sum_n |\widehat{u_j}(n)| (K_1 * |\widehat{u_j}|)(n) \\
&\leq C\beta^{-\frac{1}{2}} \|\widehat{u_j}\|_{\ell^2}^4 ,
\end{aligned}
$$

where again we applied Young's convolution inequality in the last step.

Combining the above estimates for (I), (II), and (III), we obtain that

$$
|D_B(\widehat{u_j})| \leq C\beta^{-\frac{1}{2}} \|\widehat{u_j}\|_{\ell^2}^4 + C\beta \|\widehat{u_j}\|_{\ell^2}^3 \sup_{m\in\mathbb{Z}} \left(\sum_{r=m-\beta}^{m+\beta} |\widehat{u_j}(r)|^2 \right)^{1/2} .
$$

Now, if $\{\widehat{u_j}\}$ vanishes, then for each fixed $\beta \geq 1$ and $\epsilon > 0$ there exists $N \in \mathbb{N}$ such that for all $j \geq N$,

$$
\sup_{m\in\mathbb{Z}} \sum_{r=m-\beta}^{m+\beta} |\widehat{u_j}(r)|^2 < \epsilon^6.
$$

In particular, for $\beta = \epsilon^{-2}$, there exists N such that for all $j \geq N$,

$$
|D_B(\widehat{u_j})| \leq C\epsilon^{1/2} \|\widehat{u_j}\|_{\ell^2}^4 + C\epsilon^{-2}(\epsilon^6)^{\frac{1}{2}} \|\widehat{u_j}\|_{\ell^2}^3 \leq C\epsilon.
$$

This shows that $\lim_{j\to\infty} D_B(\widehat{u_j}) = 0$. □

4 Proof of Theorem 2.1

We first prove part (ii) of the Theorem, which is the main part.

Fix $B > 0$, and suppose $J_{B,1} > B/\pi$. Let $\{u_j\}_{j\in\mathbb{N}}$ be a maximizing sequence in $L^2(\mathbb{T})$ for $J_{B,1}$, so that $\|u_j\|_{L^2} = 1$ for all $j \in \mathbb{N}$ and $\lim_{j\to\infty} W_B(u_j) = J_{B,1}$. We have that $\|\widehat{u_j}\|_{\ell^2}^2 = 1/(2\pi)$ for all $j \in \mathbb{N}$, so Lemma 3.2 applies with $M = 1/(2\pi)$, and asserts that there are three types of behavior that the sequence $\{\widehat{u_j}\}$ could exhibit. We claim that in the present situation, vanishing and splitting do not occur, so that only tightness is possible.

We suppose first, for the sake of contradiction, that the sequence $\{\widehat{u_j}\}$ is vanishing. Then from (3.5) we have that

$$
W(u_j) = 4\pi B \|\widehat{u_j}\|_{\ell^2}^4 - 2\pi B \|\widehat{u_j}\|_{\ell^4}^4 + D_B(u_j) \tag{4.1}
$$

for all $j \in \mathbb{N}$. On the other hand, from Lemmas 3.4 and 3.10 we have that $\|\widehat{u_j}\|_{\ell^4} \to 0$ and $D_B(u_j) \to 0$ as $j \to \infty$. Therefore, taking $j \to \infty$ in (4.1), we get that $J_{B,1} = B/\pi$, contradicting the assumption that $J_{B,1} > B/\pi$. Thus $\{\widehat{u_j}\}$ cannot vanish.

Next suppose, again for contradiction, that $\{\widehat{u_j}\}$ exhibits splitting. Let $a_j(n) = \widehat{u_j}(n)$ for $n \in \mathbb{N}$, fix $\delta > 0$, and for this δ define $\alpha \in (0, 1/(2\pi))$ and for each $j \in \mathbb{N}$ define sequences $\{b_j(n)\}_{n\in\mathbb{N}}$ and $\{c_j(n)\}_{n\in\mathbb{N}}$ as in alternative 2 of Lemma 3.2. For each $j \in \mathbb{N}$, let $v_j, w_j \in L^2(\mathbb{T})$ be such that $\widehat{v_j}(n) = b_j(n)$ and $\widehat{w_j}(n) = c_j(n)$ for all $n \in \mathbb{N}$. From Lemmas 3.7 and 3.9 we have that, for all $j \in \mathbb{N}$,

$$
|W_B(u_j) - W_B(v_j) - W_B(w_j)| \leq 8\pi B \|\hat{v}_j\|_{\ell^2}^2 \|\hat{w}_j\|_{\ell^2}^2 + C\delta^{\frac{1}{2}},
$$

where C is independent of δ and j. Therefore

$$\begin{aligned}W(u_j) &\le W(v_j) + W(w_j) + 8\pi B \|\widehat{v_j}\|_{\ell^2}^2 \ \|\widehat{w_j}\|_{\ell^2}^2 + C\delta^{\frac{1}{2}} \\ &\le \|v_j\|_{L^2}^4 J_{B,1} + \|w_j\|_{L^2}^4 J_{B,1} + 8\pi B \|\widehat{v_j}\|_{\ell^2}^2 \|\widehat{w_j}\|_{\ell^2}^2 + C\delta^{\frac{1}{2}} \\ &= 4\pi^2 \|\widehat{v_j}\|_{\ell^2}^4 J_{B,1} + 4\pi^2 \|\widehat{w_j}\|_{\ell^2}^4 J_{B,1}) + 8\pi B \|\widehat{v_j}\|_{\ell^2}^2 \|\widehat{w_j}\|_{\ell^2}^2 + C\delta^{\frac{1}{2}}.\end{aligned}$$

Recalling that $\|\widehat{v_j}\|_{\ell^2}^2 \le \alpha + \delta$ and $\|\widehat{w_j}\|_{\ell^2}^2 \le (M - \alpha) + \delta = (1/(2\pi) - \alpha) + \delta$, we obtain that

$$W(u_j) \le 4\pi^2 \left(\alpha^2 + \left(\frac{1}{2\pi} - \alpha \right)^2 \right) J_{B,1} + 8\pi B \alpha \left(\frac{1}{2\pi} - \alpha \right) + C\delta^{\frac{1}{2}} + C\delta + C\delta^2. \tag{4.2}$$

Taking the limit as $j \to \infty$ followed by the limit as $\delta \to 0$ in (4.2) results in

$$J_{B,1} \le 4\pi^2 \left(\alpha^2 + \left(\frac{1}{2\pi} - \alpha \right)^2 \right) J_{B,1} + 8\pi B \alpha \left(\frac{1}{2\pi} - \alpha \right). \tag{4.3}$$

But since $0 < \alpha < \frac{1}{2\pi}$, the inequality (4.3) implies that $J_{B,1} \le B/\pi$, again contradicting the assumption that $J_{B,1} > B/\pi$. Therefore $\{\widehat{u_j}\}$ can not split, either.

By Lemma 3.2, the only remaining possibility for $\{\widehat{u_j}\}$ is that one of its subsequences, when suitably translated, is tight. In other words, denoting this subsequence again by $\{\widehat{u_j}\}$, we can assert the existence of integers $m_1, m_2, m_3, \ldots$ such that for each $\epsilon > 0$, there exists an integer $r = r(\epsilon) > 0$ with the property that

$$\sum_{n=m_j-r}^{m_j+r} |\widehat{u_j}(n)|^2 \ge \frac{1}{2\pi} - \epsilon \tag{4.4}$$

for all $j \in \mathbb{N}$.

Define $v_j(x) = e^{-im_j x} u_j(x)$ for $j \in \mathbb{N}$, so that $\hat{v}_j(n) = \hat{u}_j(n + m_j)$ for all $n \in \mathbb{Z}$. By Lemma 3.1 $\{v_j\}$ is also a maximizing sequence for $J_{B,1}$. Also, from (4.4) we have that for each $\epsilon > 0$, there exists an integer $r > 0$ with the property that for all $j \in \mathbb{N}$,

$$\sum_{n=-r}^{r} |\widehat{v_j}(n)|^2 \ge \frac{1}{2\pi} - \epsilon.$$

Since the sequence $\{v_j\}_{j\in\mathbb{N}}$ is bounded in $L^2(\mathbb{T})$, with $\|v_j\|_{L^2} = 1$ for all j, there exists a subsequence, still denoted by $\{v_j\}$, that converges weakly to some function $u_0 \in L^2(\mathbb{T})$ with $\|u_0\|_{L^2} \le 1$.

We claim that in fact $\|u_0\|_{L^2} = 1$. To prove this, we start by fixing an arbitrary $k \in \mathbb{N}$. Let $\epsilon_k = \frac{1}{k}$ and choose $r_k = r(\epsilon_k) = r(\frac{1}{k})$. We define $\mu_k : \mathbb{Z} \to \{0, 1\}$ by setting $\mu_k(n) = 1$ for $|n| \le r_k$ and $\mu_r(n) = 0$ for $|n| > r_k$; and then define the low- and high-frequency components $v_{j,k}^{(l)}$ and $v_{j,k}^{(h)}$ of v_j by setting

$$\mathcal{F}\left(v_{j,k}^{(l)}\right)[n] = \mu_k(n)\widehat{v_j}(n)$$

and

$$\mathcal{F}\left(v_{j,k}^{(h)}\right)[n] = (1 - \mu_k(n))\,\widehat{v_j}(n)$$

for all $n \in \mathbb{Z}$.

We then have

$$\|v_{j,k}^{(l)}\|_{H^1}^2 = \sum_n (1 + |n|^2)|\mu_k(n)\widehat{v_j}(n)|^2 \le (1 + 4r_k^2)\|\widehat{v_j}\|_{\ell^2}^2 = \frac{1}{2\pi}\left(1 + 4r_k^2\right) \tag{4.5}$$

and

$$\|v_{j,k}^{(h)}\|_{L^2}^2 = 2\pi \sum_{|n|>r_k} |\widehat{v_j}(n)|^2 = 2\pi \left(\frac{1}{2\pi} - \sum_{|n|\le r_k} |\widehat{v_j}(n)|^2 \right) \le 2\pi\epsilon_k. \tag{4.6}$$

Siince (4.5) bounds $\{v_{j,k}^{(l)}\}$ in H^1 norm and (4.6) bounds $\{v_{j,k}^{(h)}\}$ in L^2 norm, we can assume (by passing to subsequences if necessary) that $\{v_{j,k}^{(l)}\}_{j\in\mathbb{N}}$ converges weakly in $H^1(\mathbb{T})$ to some limit $u_k^{(l)} \in H^1(\mathbb{T})$, and $\{v_{j,k}^{(h)}\}_{j\in\mathbb{N}}$ converges weakly in L^2 to some limit $u_k^{(h)} \in L^2(\mathbb{T})$ with $\|u_k^{(h)}\|_{L^2} \leq \sqrt{2\pi\epsilon_k}$. We must then have $u_0 = u_k^{(l)} + u_k^{(h)}$.

By Rellich's Lemma, the inclusion of $H^1(\mathbb{T})$ into $L^2(\mathbb{T})$ is compact. Therefore, again by passing to a subsequence, we can assume that $\{v_{j,k}^{(l)}\}_{j\in\mathbb{N}}$ converges strongly in $L^2(\mathbb{T})$ to $u_k^{(l)}$. Hence

$$\begin{aligned}
\|u_0\|_{L^2(\mathbb{T})} &= \|u_k^{(l)} + u_k^{(h)}\|_{L^2(\mathbb{T})} \\
&\geq \|u_k^{(l)}\|_{L^2(\mathbb{T})} - \|u_k^{(h)}\|_{L^2(\mathbb{T})} \\
&\geq \lim_{j\to\infty} \|v_{j,k}^{(l)}\|_{L^2(\mathbb{T})} - \sqrt{2\pi\epsilon_k} \\
&\geq \liminf_{j\to\infty} \left[\|v_j\|_{L^2(\mathbb{T})} - \|v_{j,k}^{(h)}\|_{L^2(\mathbb{T})}\right] - \sqrt{2\pi\epsilon_k} \\
&\geq \lim_{j\to\infty} \|v_j\|_{L^2(\mathbb{T})} - 2\sqrt{2\pi\epsilon_k} \\
&= 1 - 2\sqrt{2\pi\epsilon_k}.
\end{aligned}$$

We have thus proved that $\|u_0\|_{L^2} \geq 1 - 2\sqrt{2\pi\epsilon_k}$ for every $k \in \mathbb{N}$, and so we have shown that $\|u_0\|_{L^2} = 1 = \lim_{j\to\infty} \|v_j\|_{L^2}$. This is enough to conclude that $\{v_j\}$ converges to u_0 not only weakly, but also in the norm of $L^2(\mathbb{T})$. Since, as noted in Lemma 3.8, the map W_B is continuous on $L^2(\mathbb{T})$, it follows that u_0 is a maximizer for $J_{B,1}$. This completes the proof of part (ii) of Theorem 2.1.

To prove part (i) of the Theorem, let $\{u_j\}_{j\in\mathbb{N}}$ be any sequence such that $\|u_j\|_{L^2(\mathbb{T})} = \sqrt{2\pi}\|\widehat{u_j}\|_{\ell^2} = 1$ for all $j \in \mathbb{N}$ and $\{\widehat{u_j}\}$ vanishes, in the sense of Lemma 3.2. For example, we could define u_j by requiring that

$$\widehat{u_j}(n) = \begin{cases} \frac{1}{\sqrt{2\pi(2j+1)}} & \text{for } |j| \leq n \\ 0 & \text{for } |j| > n. \end{cases}$$

Since $\{\widehat{u_j}\}$ vanishes, it follows from Lemmas 3.4 and 3.10 and equation (3.5) that

$$\lim_{j\to\infty} W_B(u_j) = B/\pi, \tag{4.7}$$

and therefore we must have $J_{B,1} \geq B/\pi$.

For part (iii) of the Theorem, assume that $J_{B,1} = B/\pi$, and take $\{u_j\}$ to be any sequence such that $\|u_j\|_{L^2(\mathbb{T})} = 1$ for all $j \in \mathbb{N}$ and $\{\widehat{u_j}\}$ vanishes. As in the preceding paragraph, we have that (4.7) holds, which means that $\{u_j\}$ is a maximizing sequence. However, since $\{\widehat{u_j}\}$ vanishes, then by the remark made above in the paragraph following Lemma 3.2, it is impossible for there to exist a subsequence $\{u_{j_k}\}$ and a sequence of integers $\{m_k\}$ such that $\{\widehat{u_{j_k}}(\cdot - m_k)\}$ converges strongly in $\ell^2(\mathbb{Z})$. This then proves part (iii).

5 Existence of maximizers

In this section we give results on the set of values of $B > 0$ for which maximizers for $J_{B,1}$ exist in $L^2(\mathbb{T})$.

For what follows, it will be useful to define the map $A_B : L^2(\mathbb{T}) \to \mathbb{R}$ by

$$A_B(u) = \frac{D_B(u)}{2\pi B} - \|\hat{u}\|_{\ell^4}^4,$$

where D_B is defined in (3.6). We have the following corollary of Theorem 2.1.

Corollary 5.1. *Let $B > 0$ be given.*

(i) Suppose there exists some $w \in L^2(\mathbb{T})$ such that $A_B(w) > 0$. Then $J_{B,1} > B/\pi$, and there exists a maximizer for $J_{B,1}$ in $L^2(\mathbb{T})$.

(ii) If, on the other hand, one has that $A_B(u) < 0$ for all $u \in L^2(\mathbb{T})$, then $J_{B,1} = B/\pi$, and there do not exist any maximizers for $J_{B,1}$ in $L^2(\mathbb{T})$.

Proof. By Theorem 2.1, to prove part (i) it is enough to show that $J_{B,1} > B/\pi$ holds if and only if there exists $w \in L^2(\mathbb{T})$ such that $A_B(w) > 0$. Indeed, because $D(\lambda w) = \lambda^4 w$ for all $\lambda > 0$ and all $w \in L^2(\mathbb{T})$, we have that $A_B(w) > 0$ for some $w \in L^2(\mathbb{T})$ if and only if $A_B(w) > 0$ for some $w \in L^2(\mathbb{T})$ with $\|w\|_{L^2} = 1$. By (3.5), this is equivalent to saying that $W_B(w) > B/\pi$ for some w with $\|w\|_{L^2} = 1$. This in turn is clearly equivalent to the assertion that $J_{B,1} > B/\pi$.

To prove part (ii), note that if $A_B(u) < 0$ for all $u \in L^2(\mathbb{T})$, then from (3.5) it follows that $W_B(u) < B/\pi$ for all $u \in L^2(\mathbb{T})$ such that $\|u\|_{L^2} = 1$. In particular, $J_{B,1} \leq B/\pi$. On the other hand, from part (i) of Theorem 2.1 we have that $J_{B,1} \geq B/\pi$. Therefore, we must have $J_{B,1} = B/\pi$, and moreover there cannot exist any $u_0 \in L^2(\mathbb{T})$ such that $\|u_0\|_{L^2} = 1$ and $W_B(u_0) = J_{B,1}$. □

For $u \in L^2(\mathbb{T})$ and $p, l \in \mathbb{Z}$, define

$$a_{p,l}(u) = \sum_{n\in\mathbb{N}} \hat{u}(n)\bar{\hat{u}}(n-l)\bar{\hat{u}}(n-p)\hat{u}(n-p-l) \tag{5.1}$$

and

$$b_{p,l} = \frac{1}{B}\int_0^B e^{-2ilpt}\, dt, \tag{5.2}$$

so that from (3.6) we have

$$D_B(u) = 2\pi B \sum_{l\neq 0}\sum_{p\neq 0} a_{p,l}(u) b_{p,l}. \tag{5.3}$$

Lemma 5.2. *For all $B > 0$ and $u \in L^2(\mathbb{T})$,*

$$A_B(u) = 4\Re\left(\sum_{p=1}^{\infty} a_{p,p}(u) b_{p,p} + 2\sum_{p=1}^{\infty}\sum_{l=1}^{p-1} a_{p,l}(u) b_{p,l}\right) - a_{0,0}(u), \tag{5.4}$$

where $\Re z$ denotes the real part of the complex number z.

In particular, if the Fourier coefficients $\hat{u}(n)$ are real-valued for all $n \in \mathbb{Z}$, we have

$$A_B(u) = 4\sum_{p=1}^{\infty} a_{p,p}(u)\frac{\sin(2p^2B)}{2p^2B} + 8\sum_{p=2}^{\infty}\sum_{l=1}^{p-1} a_{p,l}(u)\frac{\sin(2plB)}{2plB} - a_{0,0}(u). \tag{5.5}$$

Proof. It is easy to see from (5.1) and (5.2) that for all $u \in L^2(\mathbb{T})$ and all p and l in $\mathbb{Z}$, we have

$$a_{p,l}(u) = a_{l,p}(u) = a_{p,-l}(u) = \overline{a_{-p,l}(u)}$$

and

$$b_{p,l} = b_{l,p} = b_{-p,-l} = \overline{b_{-p,l}}.$$

In view of these identities, the statements in the Lemma follow from (5.3) and the fact that

$$a_{0,0}(u) = \|\hat{u}\|_{\ell^4}^4.$$

□

An immediate consequence is the following nonexistence result.

Corollary 5.3. *If $B = N\pi$ for $N \in \mathbb{N}$, then $J_{B,1} = N$, and there do not exist any maximizers for $J_{B,1}$ in $L^2(\mathbb{T})$.*

Proof. From (5.2) we see that if $B > 0$ is an integer multiple of π, then $b_{l,p} = 0$ for all integers l and p such that $lp \neq 0$. Therefore, by (5.4),

$$A_B(u) = -a_{0,0}(u) = -\|\hat{u}\|_{\ell^4}^4 < 0$$

for all $u \in L^2(\mathbb{T})$. The result then follows from part (ii) of Corollary 5.1. □

To obtain existence results, we consider different test functions for w. First, define $w_1 \in L^2(\mathbb{T})$ by setting

$$\widehat{w_1}(n) = \begin{cases} 1 & \text{for } n = 0 \\ r & \text{for } n = \pm 1 \\ 0 & \text{for } |n| \geq 2, \end{cases}$$

where $r \in \mathbb{R}$. Clearly, when $w = w_1$ we have that $a_{1,1} = r^2$ and $a_{p,l} = 0$ for $(p, l) \neq (1, 1)$. Therefore, from (5.5) we get that

$$A_B(w_1) = 4r^2 \, \frac{\sin 2B}{2B} - (1 + 2r^4).$$

Since the function $f(r) = \dfrac{1 + 2r^4}{4r^2}$ has a minimum value of $\frac{\sqrt{2}}{2}$ at $r = 2^{-\frac{1}{4}}$, then there will exist a choice of $r \in \mathbb{R}$ for which $A_B(w_1) > 0$, provided that

$$\frac{\sin 2B}{2B} < \frac{\sqrt{2}}{2}. \tag{5.6}$$

Thus we see that there exists $w_1 \in L^2(\mathbb{R})$ for which $A_B(w_1) > 0$, provided $B \in (0, B_0)$, where $B_0 \approx 0.6958$ is the positive solution of $(\sin 2B_0)/2B_0 = \sqrt{2}/2$.

Next, define $w_2 \in L^2(\mathbb{T})$ by setting

$$\widehat{w_2}(n) = \begin{cases} 1 & \text{for } n = 0 \\ r & \text{for } n = \pm 1 \\ s & \text{for } n = \pm 2 \\ 0 & \text{for } |n| \geq 3, \end{cases}$$

where $r, s \in \mathbb{R}$. Here we see that the only nonzero values of $a_{p,l}(w_2)$ which appear on the right-hand side of (5.5) when $u = w_2$ are

$$\begin{aligned} a_{0,0}(w_2) &= 1 + 2r^4 + 2s^4 \\ a_{1,1}(w_2) &= r^2(1 + 2s) \\ a_{2,1}(w_2) &= 2r^2 s \\ a_{3,1}(w_2) &= r^2 s^2 \\ a_{2,2}(w_2) &= s^2. \end{aligned} \tag{5.7}$$

Therefore (5.5) gives

$$A_B(w_2) = 4r^2(1+2s)\left(\frac{\sin 2B}{2B}\right) + 16r^2 s\left(\frac{\sin 4B}{4B}\right) + 8r^2 s^2\left(\frac{\sin 6B}{6B}\right) + 4s^2\left(\frac{\sin 8B}{8B}\right) - (1+2r^4+2s^4).$$

Computations with Mathematica indicate that $\max\{A_B(w_2) : (r, s) \in \mathbb{R}^2\}$ is positive for all B such that $0 < B < B_1$ where $B_1 = 0.919 \pm .001$.

In fact, if we define $w_3 \in L^2(\mathbb{T})$ by setting

$$\widehat{w_3}(n) = \begin{cases} 1 & \text{for } n = 0 \\ p + iq & \text{for } n = \pm 1 \\ 0 & \text{for } n \geq 2, \end{cases}$$

then computations with Mathematica show that $\max\{A_B(w_3) : (p, q) \in \mathbb{R}^2\}$ is positive for all B in the interval $0 < B < B_3$, where $B_3 = 1.39 \pm .01$. For B near B_3, the maximum occurs near $p = 0.6$ and $q = 0.5$.

We can go a bit further by defining $w_4 \in L^2(\mathbb{T})$ by

$$\widehat{w_4}(n) = \begin{cases} 1 & \text{for } n = 0 \\ p + iq & \text{for } n = \pm 1 \\ p + iq & \text{for } n = \pm 2 \\ 0 & \text{for } |n| \geq 3. \end{cases}$$

Then computations with Mathematica show that $\max\{A_B(w_4) : (p,q) \in \mathbb{R}^2\}$ is positive for all B in the interval $0 < B < B_4$, where $B_4 = 2.60 \pm .01$. For B near B_4, the maximum is attained near $p = 0.7$ and $q = 0.6$.

From these computations and Corollary 5.1 we then obtain the following existence result:

Corollary 5.4. *There exist maximizers for $J_{B,1}$ in $L^2(\mathbb{T})$ for all B in the interval $0 < B < B_4$, where $B_4 = 2.60 \pm -.01$.*

6 Stability of sets of ground-state solutions of the periodic DMNLS equation

As mentioned in the introduction, the periodic DMNLS equation for functions of period L in x takes the form

$$u_t = -i\nabla H_L(u) \tag{6.1}$$

where

$$H_L(u) = -\frac{2\pi}{L}\int_0^L \int_0^1 |T_t^L u(x)|^4 \, dt \, dx.$$

The operator T_t^L is defined as a Fourier multiplier operator on $L^2_{\text{per}}(0,L)$ by setting

$$\mathcal{F}_L(T_t^L u)[n] = e^{-i(2\pi n/L)^2 t}\mathcal{F}_L u[n]$$

for all $n \in \mathbb{Z}$, where $\mathcal{F}_L$ denotes the Fourier transform on $L^2_{\text{per}}(0,L)$ (see Section 2 for notation). In particular, from (2.2) we see that $T_t = T_t^{2\pi}$.

Equation (6.1) is globally well-posed in $L^2_{\text{per}}(0,L)$, in the sense that for every $u_0(x) \in L^2_{\text{per}}(0,L)$, there is a unique strong solution $u(x,t)$ of (6.1) in $L^2_{\text{per}}(0,L)$ with $u(x,0) = u(x)$. Moreover, $H_L(u)$ and $P(u) := \frac{1}{2}\int_0^L |u|^2 \, dx$ are conserved quantities for such solutions. (See [3] for details.)

A solution of (6.1) of the form

$$u(x,t) = e^{i\omega t}\phi(x), \tag{6.2}$$

where $\phi \in L^2_{\text{per}}(0,L)$, is called a bound-state solution with profile function ϕ. Substituting into (6.1), we see that $\phi \in L^2_{\text{per}}(0,L)$ is the profile function of a bound-state solution if and only if ϕ is satisfies the equation

$$\nabla H_L(\phi) = \omega\phi \tag{6.3}$$

for some $\omega \in \mathbb{R}$.

Note that (6.3) is the Euler-Lagrange equation for the variational problem of minimizing $H_L(u)$ subject to the constraint that $P(u)$ be held constant, with ω playing the role of the Lagrange multiplier. Thus profile functions for bound-state solutions may be characterized as critical points of the variational problem. If a non-zero bound-state profile ϕ is actually a minimizer for the variational problem, then we say that the bound-state solution is a ground-state solution. That is, a bound-state solution (6.2) is a ground-state solution if $H(\phi) \le H(\psi)$ for all $\psi \in L^2_{\text{per}}(0,L)$ such that $P(\psi) = P(\phi) > 0$.

For given $\lambda > 0$ and $L > 0$, we define $S_{L,\lambda}$ to be the set of all minimizers for $H_L(u)$ subject to the constraint $P(u) = \lambda$. (Note that it may happen that no such minimizers exist, in which case $S_{L,\lambda}$ is empty.) Thus, every element of $S_{L,\lambda}$ is a ground-state solution profile; and conversely every ground-state profile belongs to $S_{L,\lambda}$ for some $\lambda > 0$. Because $H_L(u)$ and $P(u)$ are invariant under translations and under the action of multiplication by $e^{i\theta}$ for $\theta \in \mathbb{R}$, then $S_{L,\lambda}$ is also invariant under these operations. That is, if $\phi \in S_{L,\lambda}$, then $e^{i\theta}\phi(x + x_0)$ is also in $S_{L,\lambda}$ for every $\theta \in \mathbb{R}$ and every $x_0 \in \mathbb{R}$. Another way of putting this fact is that $S_{L,\lambda}$ is invariant under translations both in x and in Fourier space.

We say that a sequence $\{u_n\}$ in $L^2_{\text{per}}(0,L)$ is a minimizing sequence for $S_{L,\lambda}$ if $P(u_n) = \lambda$ for all $n \in \mathbb{N}$, and $H_L(u_n) \to I_{L,\lambda}$ as $n \to \infty$, where

$$I_{L,\lambda} = \inf\left\{H_L(u) : u \in L^2_{\text{per}}(0,L) \quad \text{and } P(u) = \lambda\right\}.$$

We observe next that profiles in $S_{L,\lambda}$ are related via dilations to the maximizers for (1.2) discussed in the preceding sections. For $\delta > 0$, define a dilation operator M_δ on functions u with domain $\mathbb{R}$, by setting

$$(M_\delta u)(x) = u(\delta x)$$

for $x \in \mathbb{R}$.

Lemma 6.1. *Suppose $L > 0$ and $\lambda > 0$ are given, and let $\delta = L/(2\pi)$ and $B = (2\pi/L)^2$. Then $\psi \in L^2_{\mathrm{per}}(0, L)$ is in $S_{L,\lambda}$ if and only if $M_\delta(\psi) \in L^2(\mathbb{T})$ is a maximizer for $J_{B,\lambda/\delta}$.*

Proof. We have that $u \in L^2_{\mathrm{per}}(0, L)$ with $P(u) = \lambda$ if and only if $v = M_\delta u \in L^2(\mathbb{T})$ with $\|v\|^2_{L^2} = \lambda/\delta$. A calculation shows that

$$T_t(v)(x) = M_\delta \left(T^L_{\delta^2 t}(u)\right),$$

whence one obtains that

$$H_L(u) = -\frac{1}{B} W_B(v).$$

Taking the infimum over all $u \in L^2_{\mathrm{per}}(0, L)$ with $P(u) = \lambda$, or equivalently over all $v \in L^2(\mathbb{T})$ with $\|v\|^2_{L^2} = \lambda/\delta$, we obtain the desired result. □

Theorem 6.2. *Suppose $L > 2\pi/\sqrt{B_4}$, where B_4 is as defined in Corollary 5.4. Then for every $\lambda > 0$, $S_{L,\lambda}$ is nonempty, and is furthermore stable, in the following sense. For $u \in L^2_{\mathrm{per}}(0, L)$, define*

$$d(u, S_{L,\lambda}) = \inf_{\phi \in S_{L,\lambda}} \|u - \phi\|_{L^2_{\mathrm{per}}(0,L)}.$$

Then for every $\epsilon > 0$, there exists $\delta > 0$ such that if $u_0 \in L^2_{\mathrm{per}}(0, L)$ with $d(u_0, S_{L,\lambda}) < \delta$, the solution $u(x, t)$ of (6.1) with initial data $u(\cdot, 0) = u_0$ will satisfy $d(u(\cdot, t), S_{L,\lambda}) < \epsilon$ for all $t \geq 0$.

Proof. Notice that if $L > 2\pi/\sqrt{B_4}$, then $B = (2\pi/L)^2$ satisfies $0 < B < B_4$. Also, as noted above after equation (2.6), the existence of a maximizing function for $J_{B,1}$ is equivalent to the existence of a maximizing function for $J_{B,\lambda}$ for every $\lambda > 0$. Therefore it follows immediately from Lemma 6.1 and Corollary 5.4 that $S_{L,\lambda}$ is nonempty. Furthermore, from Theorem 2.1 and Lemma 6.1 it also follows that for every minimizing sequence for $S_{L,\lambda}$, one can find a subsequence which, after translations in Fourier space, converges in $L^2_{\mathrm{per}}(0, L)$ to a function in $S_{L,\lambda}$.

The stability of the set $S_{L,\lambda}$ follows from a standard argument, which we summarize here (more details, for example, can be found in [3]). Suppose, to the contrary, that the set $S_{L,\lambda}$ is not stable. Then one must be able to find some $\epsilon_0 > 0$, some sequence of initial data $\{u_{0n}\}$ in $L^2_{\mathrm{per}}(0, L)$ with corresponding solutions $\{u_n(x, t)\}$, and some sequence of times $\{t_n\}$ in $(0, \infty)$ such that $d(u_{0n}, S_{L,\lambda}) \to 0$ as $n \to \infty$ and $d(u_n(\cdot, t_n), S_{L,\lambda}) \geq \epsilon_0$ for all $n \in \mathbb{N}$. The assumption on the initial data $\{u_{0n}\}$ implies that by choosing a sequence $\{\alpha_n\}$ in $(0, \infty)$ with $\lim_{n\to\infty} \alpha_n = 1$ such that $P(\alpha_n u_{0n}) = \lambda$ for all sufficiently large n, we can obtain a minimizing sequence $\{\alpha_n u_{0n}\}$ for $S_{L,\lambda}$. Moreover, since H_L and P are conserved functionals for (6.1), $\{\alpha_n u_n(\cdot, t_n)\}$ is also a minimizing sequence for $S_{L,\lambda}$. Therefore there exists a subsequence of $\{\alpha_n u_n(\cdot, t_n)\}$ which, after translations in Fourier space, converges in $L^2_{\mathrm{per}}(0, L)$ to a function in $S_{L,\lambda}$. Since $S_{L,\lambda}$ is invariant under the action of translation in Fourier space, it follows that $d(\alpha_n u_n(\cdot, t_n), S_{L,\lambda})$, and hence also $d(u_n(\cdot, t_n), S_{L,\lambda})$, converges to zero as $n \to \infty$. But this contradicts the assertion that $d(u_n(\cdot, t_n), S_{L,\lambda}) \geq \epsilon_0$ for all $n \in \mathbb{N}$.

□

We remark that similar results on the stability of sets of ground-state solutions of the nonlinear Schrödinger equation $iu_t + u_{xx} + |u|^p u_x = 0$ date back to the work of Cazenave and Lions in [11]. In fact, for the nonlinear Schrödinger equation, Cazenave and Lions prove a stronger form of stability called *orbital stability*: namely, they show for a given ground-state profile, the two-dimensional set $\{e^{i\theta}\phi(x + x_0) : \theta \in \mathbb{R}, x_0 \in \mathbb{R}\}$ is stable in the above sense. (Note that the term "orbital stability" is slightly inaccurate here, in that the orbit in the usual sense of the ground-state solution would be the one-dimensional set $\{e^{i\theta}\phi(x) : \theta \in \mathbb{R}\}$. It is easy to see, however, that this one-dimensional set is not stable in the above sense, cf. Remark 8.3.3 on p. 274 of [10].) In order to prove this stronger form of stability, one generally needs more information on the structure of the set of minimizers of the variational problem. In the case of the nonlinear Schrödinger equation, it follows from the elementary theory of ordinary differential equations that the ground-state profile for a given L^2 norm is unique up to translations and multiplications by phase shifts $e^{i\theta}$, which allows one to deduce orbital stability. However, no such uniqueness result is available yet for the DMNLS equation.

In light of the fact that ground-state solutions for the nonlinear Schrodinger equation have, up to symmetries, profiles that are real-valued even functions of x, it is interesting to note that at least for some values of B, ground-state solutions of the periodic DMNLS equation cannot have real-valued even profiles:

Corollary 6.3. *In the case $L = \sqrt{8\pi}$, the set $S_{L,\lambda}$ of ground-state profiles is nonempty for every $\lambda > 0$. However, none of the the functions in $S_{L,\lambda}$ are real-valued and even.*

Proof. The assertion that $S_{\sqrt{8\pi},\lambda}$ is nonempty follows from Corollary 5.4 and Lemma 6.1.

Suppose now that $\psi \in S_{\sqrt{8\pi},\lambda}$. Then from Lemma 6.1 we have that $v = M_\delta \psi \in L^2(\mathbb{T})$ is a maximizer for $J_{B,\lambda/\delta}$, where $B = \pi/2$ and $\delta = \sqrt{2/\pi}$. From (5.5) we see that in the case $B = \pi/2$, for every function $u \in L^2(\mathbb{T})$ such that $\hat{u}(n)$ is real-valued for all $n \in \mathbb{N}$, we have $A_B(u) = -a_{0,0}(u) < 0$. On the other hand, from Corollary 5.1 and its proof one sees that for a maximizer v for $J_{B,\lambda/\delta}$, one necessarily has $A_B(v) > 0$. Therefore the Fourier coefficients of v cannot be real-valued. Since real-valued even functions must have real-valued Fourier coefficients, it follows that v cannot be real-valued and even. Therefore ψ cannot be real-valued and even either. □

We conclude with an easy nonexistence result, which shows in particular that Theorem 6.2 cannot be extended to all positive values of L.

Theorem 6.4. *If $L = 2\sqrt{\pi/N}$ for some $N \in \mathbb{N}$, then $S_{L,\lambda}$ is empty for every $\lambda > 0$. Hence, for these values of L, the periodic DMNLS equation (6.1) has no ground-state solutions.*

Proof. Suppose $L = 2\sqrt{\pi/N}$ for some $N \in \mathbb{N}$, and $\lambda > 0$. From Lemma 6.1, we see that a function $\psi \in L^2_{\text{per}}(0, L)$ can be in $S_{L,\lambda}$ only if $M_\delta(\psi) \in L^2(\mathbb{T})$ is a maximizer for $J_{2\pi,\lambda/\delta}$, where $\delta = 1/\sqrt{N\pi}$. But from Corollary 5.3 we know that $J_{N\pi,1}$, and hence also $J_{N\pi,\lambda/\delta}$, can have no maximizers. Therefore $S_{L,\lambda}$ must be empty. □

References

[1] M. Ablowitz, *Nonlinear Dispersive Waves: Asymptotic Analysis and Solitons*, Cambridge, 2011.

[2] M. Ablowitz and G. Biondini, Multiscale pulse dynamics in communication systems with strong dispersion management, Optics Letters 23 (1998), 1668–1670.

[3] O. Adekoya, Periodic solutions of the dispersion-managed nonlinear Schrödinger equation, Ph.D. thesis, University of Oklahoma, 2019.

[4] J. Albert and E. Kahlil, On the well-posedness of the Cauchy problem for some nonlocal nonlinear Schrödinger equations, Nonlinearity 30 (2017), 2308–2333.

[5] J. Bennett, N. Bez, A. Carbery, and D. Hundertmark, Heat-flow monotonicity of Strichartz norms. Anal. PDE 2 (2009), 147–158.

[6] J. Bourgain, Fourier transform restriction phenomena for certain lattice subsets and applications to nonlinear evolution equations Part 1: Schrödinger equations, Geom. Funct. Anal. 3 (1993), 107–156.

[7] E. Carneiro, A sharp inequality for the Strichartz norm, arXiv:0809.4054.

[8] E. Carneiro and D. Oliveira e Silva, IMRN (2015), 8233–8267.

[9] E. Carneiro, D. Oliveira e Silva, M. Sousa, and B. Stovall, Extremizers for adjoint Fourier restriction on hyperboloids: the higher-dimensional case. Indiana Univ. Math. J. 70 (2021), 535–559.

[10] T. Cazenave, *Semilinear Schrödinger Equations*, Amer. Math. Soc., 2003.

[11] T. Cazenave and P. Lions, Orbital stability of standing waves for some nonlinear Schrödinger equations, Comm. Math. Phys. 85 (1982), 549–561.

[12] M. Choi, D. Hundertmark, and Y. Lee, Thresholds for existence of dispersion management solitons for general nonlinearities, SIAM J. Math. Anal. 49 (2017), 1519–1569.

[13] M. Christ and S. Shao, Existence of extremals for a Fourier restriction inequality, Anal. PDE 5 (2012), 261–312.

[14] M. Erdoğan, D. Hundertmark, and Y. Lee, Exponential decay of dispersion managed solitons for vanishing average dispersion, Math. Res. Lett. 18 (2011), 11–24.

[15] L. Fanelli, L. Vega, and N. Visciglia, On the existence of maximizers for a family of restriction theorems, Bull. Lond. Math. Soc. 43 (2011), 811–817.

[16] D. Foschi, Maximizers for the Strichartz inequality. J. Eur. Math. Soc. 9 (2007), 739–774.

[17] D. Foschi, Global maximizers for the sphere adjoint Fourier restriction inequality, J. Funct. Anal., 268 (2015), 690–702.

[18] D. Foschi and D. Oliveira e Silva, Some recent progress on sharp Fourier restriction theory, Analysis Math. 43 (2017), 241–265.

[19] R. Frank, E. Lieb, and J. Sabin, Maximizers for the Stein-Tomas inequality, Geom. Funct. Anal. 26 (2016), 1095–1134.

[20] R. Frank and J. Sabin, Extremizers for the Airy-Strichartz inequality, Math. Ann. 372 (2018) 1121–1166.

[21] I. Gabitov and S. Turitsyn, Averaged pulse dynamics in a cascaded transmission system with passive dispersion compensation, Optics Letters 21 (1996) 327–329.

[22] J. Ginibre, Le problème de Cauchy pour des EDP semi-linéaires périodiques en variables d'espace, Séminaire N. Bourbaki (1994–1995) 163–187.

[23] W. Green and D. Hundertmark, Exponential decay of dispersion-managed solitons for general dispersion profiles, Lett. Math. Phys. 106 (2016) 221–249.

[24] D. Hundertmark, P. Kunstmann, and R. Schnaubelt, Stability of dispersion managed solitons for vanishing average dispersion, Archiv der Mathematik 104 (2015), 283–288.

[25] D. Hundertmark and Y. Lee, Decay estimates and smoothness for solutions of the dispersion managed non-linear Schrödinger equation, Commun. Math. Phys. 286 (2009), 851–873.

[26] D. Hundertmark and Y. Lee, On non-local variational problems with lack of compactness related to non-linear optics, J. Nonlinear Sci. 22 (2012), 1–38.

[27] D. Hundertmark, Y. Lee, T. Ried, and V. Zharnitsky, Solitary waves in nonlocal NLS with dispersion averaged saturated nonlinearities, J. Diff. Eq. 265 (2018), 3311–3338.

[28] D. Hundertmark and V. Zharnitsky, On sharp Strichartz inequalities in low dimensions. Int. Math. Res. Not. 2006, Art. ID 34080, 18 pp.

[29] J. Jiang and S. Shao, On characterization of the sharp Strichartz inequality for the Schrödinger equation. Anal. PDE 9 (2016), 353–361.

[30] C. Kenig, G. Ponce, and L. Vega, The Cauchy problem for the Korteweg-de Vries equation in Sobolev spaces of negative indices, Duke Math. J. 71 (1993) 1–21 .

[31] M. Kunze, On the existence of a maximizer for the Strichartz inequality, Comm. Math. Phys. 243 (2003), 137–162.

[32] M. Kunze, On a variational problem with lack of compactness related to Strichartz inequality, Calc. Var. Partial Differential Equations 19 (2004) 307–336.

[33] M. Kunze, J. Moeser, and V. Zharnitsky, Ground states for the higher-order dispersion managed NLS equation in the absence of average dispersion, J. Diff. Eq. 209 (2005), 77–100.

[34] P. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case, Parts 1 and 2, Annales de l'I.H.P. Analyse non linéaire 1 (1984), no. 2 and no. 4, 109–145 and 223–283.

[35] D. Luenberger, *Optimization by Vector Space Methods*, Wiley, 1969.

[36] A. Nahmod, The nonlinear Schrodinger equation on tori: integrating harmonic analysis, geometry, and probability, Bull. Amer. Math. Soc. (N.S.) 53 (2016), 57–91.

[37] D. Oliveira e Silva and R. Quilodrán, Global maximizers for adjoint Fourier restriction inequalities on low dimensional spheres, J. Funct. Anal. 280 (2021), no. 7, Paper No. 108825, 73 pp.

[38] I. Segal, Space-time decay for solutions of wave equations, Adv. Math. 22 (1976), 305–311.

[39] S. Shao, Maximizers for the Strichartz and the Sobolev-Strichartz inequalities for the Schrödinger equation. Electron. J. Differential Equations 2009, 13 pp.

[40] S. Shao, On existence of extremizers for the Tomas-Stein inequality for S^1, J. Funct. Anal. 270 (2016), 3996–4038.

[41] M. Stanislavova, Regularity of ground state solutions of DMNLS equations, J. Diff. Eq. 210 (2005), 87–105.

[42] E. Stein, Oscillatory integrals in Fourier analysis, in *Beijing Lectures in Harmonic Analysis, Ann. of Math. Stud. 112*, Princeton Univ. Press, 1986; pp. 307–355.

[43] B. Stovall, Extremizability of Fourier restriction to the paraboloid, Adv. Math. 360 (2020), 106898, 18 pp.

[44] R. Strichartz, Restriction of Fourier transform to quadratic surfaces and decay of solutions of wave equations, Duke Math. J., 44 (1977), 705–774.

[45] T. Tao, Some recent progress on the restriction conjecture, in *Fourier Analysis and Convexity (L. Brandolini et al., eds.)*, Birkhäuser, 2004; pp. 217–243.

[46] T. Tao, *Nonlinear Dispersive Equations: Local and Global Analysis*, CBMS regional conference series in mathematics, 2006.

[47] P. Tomas, A restriction theorem for the Fourier transform, Bull. Amer. Math. Soc. 81(1975), 477–478.

[48] S. Turitsyn, B. Bale, and M. Fedoruk, Dispersion-managed solitons in fibre systems and lasers, Phys. Reports 521 (2012), 135–203.

[49] V. Zharnitsky, E. Grenier, S. Turitsyn, C. Jones, and J. Hesthaven, Ground states of dispersion managed nonlinear Schrödinger equation, Phys. Rev. E 62 (2000) 7358–7364.

[50] A. Zygmund, On Fourier coefficients and transforms of functions of two variables, Studia Math. 50 (1974), 189–201.