IMA Journal of Applied Mathematics (1991) **46**, 1–19

Total Positivity and the Stability of Internal Waves in Stratified Fluids of Finite Depth

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[Received 25 September 1989 and in revised form 4 January 1990]

Solitary internal waves in stratified fluids of finite depth are investigated. Using a general criterion for the stability of travelling-wave solutions of one-dimensional nonlinear long-wave models and ideas from the theory of totally positive operators, the stability of the solitary-wave solutions of the intermediate long-wave equation is demonstrated. In the process of carrying out this programme of research, new general results are obtained that bear upon the stability of solitary-wave solutions of a class of model equations of the Korteweg–de Vries type that govern approximately the evolution of small-amplitude long waves.

1. Introduction

THE MATHEMATICALLY exact nonlinear stability theory of solitary-wave solutions of model equations for long waves began with the appearance of Benjamin's (1972) paper on the Korteweg–de Vries and the regularized long-wave equations. Benjamin's theory, which was influenced by the old work of Boussinesq (1872, 1877) and the modern ideas of Arnol'd (1965, 1966) has seen terrific development and refinement in recent years. Not only have the proofs been simplified and the basic result sharpened, but also the range of applicability has broadened. The present paper contributes mainly to the second theme of elucidation of Benjamin's ideas by using the sharpest available set of abstract conditions that are known to imply stability together with a novel argument that relies upon the theory of totally positive operators to extend the class of model equations which are known to possess stable solitary waves.

The original observations of a surface solitary wave in a channel by Scott-Russell (1845) left little doubt about their stability and persistence. Similar conclusions are warranted regarding internal solitary waves based on laboratory observations (Walker, 1973) and to some extent various field studies (see Apel *et al.*, 1975; Farmer & Smith, 1978; Hunkins & Fliegel, 1973; Osborne & Burch, 1980; Sandstrom & Elliott, 1984). Benjamin's theory demonstrated this stability unequivocally in the context of the Korteweg–de Vries equation that governs approximately the propagation of the sort of small-amplitude long waves observed by Scott-Russell. However, it has come to light recently in the work of Bona *et al.* (1987) that solitary-wave solutions of certain model equations of the Korteweg–de Vries type need not be stable. Moreover, in some instances, numerical simulations indicate that the instability manifests itself by the formation of a singularity in the solution (Bona *et al.*, 1986, 1989). Because the alternative is known to occur, there is added interest in further studies regarding stability.

The particular physical situation to which the results to be derived presently apply is the propagation of internal waves in certain density-stratified fluids. Consider an incompressible inviscid fluid trapped between two horizontal planes lying a positive distance h apart. Let x and z denote the horizontal coordinates and y the vertical coordinate in a standard Cartesian frame. At rest, the fluid is supposed to possess a stable density stratification $\rho = \rho(y)$ which depends only upon the vertical coordinate. The effects of diffusion are ignored, this being a sensibly accurate assumption on the time scales of the typical wave motions to be considered. Because of the stratification, this system can support waves, and interest will be focused on irrotational motions that propagate in the direction of the positive x-axis and which are independent of the other horizontal coordinate z. In this situation, the governing equations are the two-dimensional Euler equations together with the stipulation that the vertical velocity component is zero at the top and bottom boundaries.

In the configuration just described, solitary waves are known to exist under a variety of reasonable assumptions about the underlying density distribution ρ, and in both the case of a rigid upper boundary and where the upper boundary is free (see Benjamin, 1966; Turner, 1981; Bona *et al.*, 1983; Amick 1984; Amick & Turner, 1986, 1989; Bona & Sachs, 1989).

However, a stability theory based on the Euler equations is not yet available, even in this two-dimensional situation. Indeed, as remarked in Bona & Sachs (1989: §6), the ideas that come to the fore in the present paper and its direct precursors appear inadequate to treat the question of stability of solitary-wave solutions of the Euler equations.

In certain special circumstances, the Euler equations may be simplified on the basis of a rational approximation procedure, and model evolution equations derived. These model equations also possess travelling-wave solutions that approximate rather well a solitary-wave solution of the Euler equations in the regime of the model's formal validity. Depending upon the parameters of the problem, the appropriate model equation may be the Korteweg–de Vries equation (the KdV equation; see Benney, 1966), the Benjamin–Ono equation (the BO equation; see Benjamin, 1967), or one of the one-parameter family of intermediate long-wave equations (the ILW equation; see Joseph, 1977, and Kubota *et al.*, 1978). A recounting of the parameter regimes that lead to the various models may be found in Redekopp (1983). It is well understood that the ILW equations converge formally to the KdV or BO equations in certain extreme limits. The mathematical theory showing that this is so is contained in Albert *et al.* (1987) and Abdelouhab *et al.* (1989). The solitary-wave solutions of KdV are known to be stable (Benjamin, 1972; Bona, 1975) as are those of the BO equation (Bennett *et al.*, 1983). Using these facts and their general theory of stability of solitary waves, Albert *et al.* (1987) demonstrated that solitary-wave solutions of the ILW equation are stable at least in the parameter range where the ILW dispersion relation is close to that of the KdV equation or, alternatively, close to that of the BO equation. This theory left unanswered the question of whether or not solitary-wave solutions of all amplitudes for the entire range of ILW equations are stable, though, considering the results in hand, it did seem safe to so conjecture. It is our purpose here to settle this issue.

The plan of the paper is as follows. The next section is devoted briefly to describing the notation that will be in force and to making a few preliminary remarks regarding general properties of equations such as the ILW evolution equations. In Section 3, we recall the existing theory of necessary conditions for stability of solitary waves. Section 4 is devoted to reducing most of the necessary conditions for stability specified by the theory to forms which are easily verified in the case of solitary-wave solutions of the ILW equation. In the process, methods from the theory of total positivity are brought to bear (see Karlin, 1964, 1968). This sets the stage for the verification of the conditions for stability in Section 5, thus leading to the principal conclusion of the paper. The Appendix contains some commentary on technical issues arising in Section 3.

2. Notation and preliminary remarks

The notation in force throughout the paper will be that which is currently standard in the theory of partial differential equations. In general, if $\mathscr{B}$ is any Banach space, the norm on $\mathscr{B}$ will be denoted $\|\cdot\|_{\mathscr{B}}$, and, if $\mathscr{B}$ is also a Hilbert space, the inner product of two elements $f, g \in \mathscr{B}$ will be denoted $(f, g)_{\mathscr{B}}$. However, for Ω an open subset of Euclidean space, the norm in the L_2-based Sobolev space $H^r(\Omega)$ will be abbreviated to simply $\|\cdot\|_r$. Similarly, the norm on the standard space $L_p(\Omega)$ will be written $|\cdot|_p$. Note that $L_2(\Omega)$ has two abbreviated notations in this scheme. (Usually, the underlying spatial domain to which these function classes are referred is the real line $\mathbb{R}$. When no specification is made, $\Omega = \mathbb{R}$ will be understood.) The inner product in $L_2(\Omega)$ of two functions f and g is written unadorned as (f, g).

The equations considered in this paper are of the form

$$u_t + u_x + (f(u))_x - (Mu)_x = 0, \tag{2.1}$$

where $f : \mathbb{R} \to \mathbb{R}$ and M is defined as a Fourier multiplier operator by

$$\widehat{(Mg)}(k) = \alpha(k)\hat{g}(k)$$

for all $k \in \mathbb{R}$. (Here and in the sequel, circumflexes will be used to indicate Fourier transforms.) The symbol $\alpha(k)$ of M is assumed to be a measurable function satisfying the conditions

$$a\,|k|^{m_1} \leq \alpha(k) \leq b(1 + |k|)^{m_2} \tag{2.2}$$

for all $k \in \mathbb{R}$, where $1 \le m_1 \le m_2$ and $a, b > 0$. For simplicity, it is assumed the function f lies in $C^\infty(\mathbb{R})$, though the theory developed below is easily adapted to the situation where f has finite regularity. In case $m_1 = 1$, we follow Bona *et al.* (1987) and make the additional assumption that f and its first two derivatives grow at most like a polynomial at infinity. That is to say, there exists $r \in \mathbb{R}$ such that $|f(x)| \le (1+|x|)^r$ for all $x \in \mathbb{R}$, with similar estimates holding for f' and f''. Without loss of generality, it may be presumed that $f(0) = f'(0) = 0$.

Primary interest will be attached to the pure initial-value problem in which a solution $u = u(x, t)$ of (2.1) is sought for $(x,t) \in \mathbb{R} \times [0, T]$ such that it has a specified initial value $u(\cdot, 0) = u_0$. Here, the parameter T is positive, and, if u extends as a solution of (2.1) to arbitrarily large values of t, we say it is a *global* solution. Otherwise, we refer to the solution as *local*.

In discussing the properties of equation (2.1), the linear space $\mathscr{X}$ composed of those $g \in L_2$ for which

$$\|g\|_{\mathscr{X}} = \left(\int_{-\infty}^{\infty} [1 + \alpha(k)]\, |\hat{g}(k)|^2 \,\mathrm{d}k\right)^{\frac{1}{2}} < +\infty$$

arises naturally. Its dual $\mathscr{X}^*$ may be realized as the space of all tempered distributions V whose Fourier transform $\hat{V}$ is given by a measurable function for which

$$\|V\|_{\mathscr{X}^*} = \left(\int_{-\infty}^{\infty} \frac{|\hat{V}(k)|^2}{1 + \alpha(k)} \,\mathrm{d}k\right)^{\frac{1}{2}} < +\infty.$$

The pairing between $\mathscr{X}$ and $\mathscr{X}^*$ will find use below; if $f \in \mathscr{X}$ and $V \in \mathscr{X}^*$, then we will write $\langle V, f\rangle$ for $V(f)$. Notice that, if V happens to be given by an L_2 function g, then, as follows immediately from the concrete representation of $\mathscr{X}^*$ mentioned above and Plancherel's theorem, $\langle f, g\rangle = (f, g)$, the L_2 inner product.

The local well-posedness of the initial-value problem for various forms of (2.1) has been studied by Kato (1983), Iorio (1986), and Abdelouhab *et al.* (1989). For the purposes of this paper, the following result will suffice.

THEOREM 1 *Suppose* $s > \frac{3}{2}$, *let* M *be as described above with* (2.2) *in force, and assume that* f *lies in the Hölder space* $C^{s+1}(\mathbb{R})$. *For each* $u_0 \in H^s$, *there exists* $T^* > 0$ *depending only on* $\|u_0\|_s$ *such that* (2.1) *has a unique solution* u *in* $C([0, T^*); H^s)$ with $u(\cdot, 0) = u_0$. For any $T < T^*$, *the map that associates to* u_0 *in* H^s *the unique solution* u *in* $C([0, T]; H^s)$ *is continuous. For all* $T < T^*$ *and* j *such that* $s - j(m_2 + 1) > -\frac{1}{2} - m_2$, *it follows that* $\partial_t^j u$ *lies in* $C([0, T]; H^{s-j(m_2+1)})$. *Moreover, for* $T < T^*$, *the map that associates the solution* u *to the initial data* u_0 *is continuous from* H^s *to* $\cap_j C^j([0, T]; H^{s-j(m_2+1)})$, *where the intersection is taken over those* j *for which* $s - j(m_2 + 1) > -\frac{1}{2} - m_2$.

Remark. In many special circumstances the initial-value problem for (2.1) is globally well-posed, meaning that for each u_0 one may choose $T^* = +\infty$. A definitive theory for when this is so is not yet available, but the papers of Kato (1983) and Albert *et al.* (1988) contain suggested sufficient conditions, which the numerical simulations of Bona *et al.* (1986) indicate are sharp.

3. Sufficient conditions for stability

In this section, conditions are introduced that imply the stability of solitary-wave solutions of the model equation (2.1). These will be used in Sections 4 and 5 to prove stability of solitary-wave solutions of the ILW equation. The material in this section is not new, having appeared in various forms in the works of several authors (see Weinstein, 1983, 1985, 1986; Albert *et al.*, 1987; Bona *et al.*, 1987; Grillakis *et al.*, 1987). Consequently, the present discussion will be limited to stating a convenient form for the basic stability theorem and providing some commentary on the proof.

To begin, define a *solitary-wave solution* of (2.1) to be a travelling wave of the form $u(x, t) = \varphi(x - Ct)$, where $\varphi \in \mathscr{X}$, the constant C is a positive real number larger than one, and the solitary-wave profile $\varphi = \varphi_C$ is then a solution of the equation

$$M\varphi' + [(C-1) - f'(\varphi)]\varphi' = 0. \tag{3.1a}$$

Integrating (3.1a) once, and imposing zero boundary conditions at infinity, we see that φ also satisfies the equation

$$M\varphi + (C-1)\varphi - f(\varphi) = 0. \tag{3.1b}$$

Originally, one interprets (3.1) to hold in the sense of distributions, but provided f is a C^∞ function, a simple bootstrap argument shows that, if $\varphi \in \mathscr{X}$ is a distributional solution of (3.1b), then in fact $\varphi \in H^\infty$ (φ and all its derivatives lie in L_2) and so φ is a classical solution of (3.1). Even if f has finite regularity, say $f \in C^k$ where $k \geqslant 1$, we can still infer that φ lies at least in H^{m_1+k}.

It is typically the case, and will be assumed here, that (3.1) has a solution $\varphi = \varphi_C$ for each value of $C > 1$ and that the correspondence $C \mapsto \varphi_C$ is a C^1 map from $(1, \infty)$ into L_2. (The latter assumption is not strictly necessary in what follows, but it is quite convenient and appears to obtain in many interesting cases.) For general results on the existence of solutions of (3.1), the reader may consult Weinstein (1987) or Benjamin *et al.* (1990).

In studying the stability of a solitary wave $\varphi = \varphi_C$, one considers the associated linear operator L defined by

$$Lg(x) = Mg(x) + [(C-1) - f'(\varphi(x))]g(x). \tag{3.2}$$

As explained in Albert *et al.* (1987: Prop. 1), L is a self-adjoint unbounded operator on L_2 whose continuous spectrum is the interval $[(C-1), \infty)$, and the remainder of its spectrum is a finite number of isolated eigenvalues located on the real axis to the left of $C - 1$. In particular, (3.1a) shows that zero is an eigenvalue of L with eigenfunction φ'.

For any $r \in \mathbb{R}$ and any function whose domain is $\mathbb{R}$, define $\tau_r(\varphi)$ to be the translation of φ by r, namely,

$$\tau_r\varphi(x) = \varphi(x + r).$$

Following Bona *et al.* (1987), for any $\eta > 0$, define the set $\mathscr{U}_\eta$ by

$$\mathscr{U}_\eta = \{y : y \in \mathscr{X} \text{ and } \inf_{r \in \mathbb{R}} \|y - \tau_r\varphi\|_{\mathscr{X}} < \eta\}.$$

Then φ is said to be a *stable* solitary wave if, for any $\varepsilon > 0$, there exists a $\delta > 0$ and an $s > \frac{3}{2}$ such that, for $u_0 \in \mathcal{U}_\delta \cap \mathrm{H}^s$, the solution u of (2.1) with $u(x, 0) = u_0$ is global and satisfies $u(\bullet, t) \in \mathcal{U}_\varepsilon$ for all $t > 0$.

THEOREM 2 *Fix $C_0 > 1$ and suppose that $\varphi = \varphi_{C_0}$ is a solitary-wave solution of* (2.1) *possessing the following properties*:

(P1) L has a simple negative eigenvalue λ;
(P2) L has no negative eigenvalue other than λ;
(P3) the eigenvalue 0 of L is simple;
(P4) $(\mathrm{d}/\mathrm{d}C)(\int_{-\infty}^{\infty} (\varphi_C)^2 \,\mathrm{d}x) > 0$ at $C = C_0$.

Suppose also that, for some $s > \frac{3}{2}$, the initial-value problem

$$\begin{aligned} &u_t + u_x + f(u)_x - Mu_x = 0 \quad &&\text{for } x \in \mathbb{R},\ t \geq 0 \\ &u(x, 0) = u_0(x) &&\text{for } x \in \mathbb{R} \end{aligned}$$

is globally well-posed in the sense of Theorem 1, at least for initial data u_0 lying simultaneously in H^s and in some neighbourhood in $\mathscr{X}$ of φ. Then φ is stable.

Remarks. A commentary on the proof of Theorem 2 is provided in the Appendix. The conclusion of Theorem 2 remains valid if (P4) is replaced by the condition

(P4′) for any $\psi \in \mathscr{X}$, if $L\psi = \varphi$, then $(\psi, \varphi) < 0$.

A simple computation that involves differentiating equation (3.1) with respect to the variable C shows that conditions (P3) and (P4) together imply (P4′). Condition (P4′) is of interest in a situation where one does not have a family of solitary waves $\{\varphi_C\}$ depending smoothly on C.

The notion of stability enunciated above entails that solutions of (2.1) which are global in time emanate from initial data sufficiently near to the stable solitary wave in question. While this property has been assumed in the statement of Theorem 2, it is often the case that local existence theories such as that mentioned in Theorem 1 can be applied iteratively to extend interesting classes of solutions smoothly to arbitrarily large times by use of additional *a priori* bounds, thus providing a global existence theory. Even in the event that such bounds are not available with respect to solutions corresponding to general initial data, the stability theory itself may provide them for solutions that initially resemble a solitary wave. An example wherein exactly this situation obtains is for the Boussinesq-type equations studied by Bona & Sachs (1988).

In the next section, attention is given to providing easily verifiable conditions that imply the hypotheses of Theorem 2.

4. Spectral analysis of the operator L

In practice, it may be quite difficult to determine explicitly the spectrum of the operator L associated with a given solitary wave φ. Therefore it is desirable to find general conditions on φ, M, and f which imply that the conditions in the hypotheses of Theorem 2 are satisfied. In Albert *et al.* (1987), it was shown that property (P1) is a consequence of positivity conditions on the operator M and the solitary wave φ. This result is restated below as Theorem 3. The main result of this section is Theorem 4, which shows that property (P3) is a consequence of a 'second-order' positivity condition on the solitary wave φ to be spelled out presently.

For any $\mu > 0$, define the function $K_\mu(x)$ by the formula $(K_\mu)(k) = [\mu + \alpha(k)]^{-1}$. Notice that, as a consequence of the assumption $\alpha(k) \geq a\,|k|^{m_1}$ with $m_1 \geq 1$, the function $K_\mu(x)$ is well-defined in L_2.

THEOREM 3 *Suppose $K_\mu(x) > 0$ for all $\mu > 0$ and $x \in \mathbb{R}$, and suppose $\varphi(x) > 0$ for $x \in \mathbb{R}$. Then property* (P1) *holds for L.*

Next, for a given solitary-wave solution $\varphi \in \mathscr{X}$ satisfying (3.1), define a function $K(x)$ by

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ixz} f'(\varphi(z))\,dz.$$

As noted in Section 3, φ is in H^∞ and hence $\varphi \in L_\infty$. Because $f \in C^2$ and $f'(0) = 0$, one can infer the existence of a constant A such that $|f'(\varphi(x))| \leq A\,|\varphi(x)|$ for all $x \in \mathbb{R}$. It follows that $f'(\varphi) \in L_2$, and so $K(x)$ is well-defined as a function in L_2.

In the remainder of this section, $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ will be used to denote elements of $\mathbb{R}^2$. Let $\Delta = \{\bar{x} \in \mathbb{R}^2 : x_1 < x_2\}$, and define $K_2(\bar{x}, \bar{y})$ for $\bar{x}, \bar{y} \in \Delta$ by

$$K_2(\bar{x}, \bar{y}) = K(x_1 - y_1)K(x_2 - y_2) - K(x_1 - y_2)K(x_2 - y_1).$$

THEOREM 4 *Suppose that α and φ are even functions, that*

(i) $K(x) > 0$ *and* $\hat{\varphi}(x) > 0$ *for all* $x \in \mathbb{R}$, *and that*
(ii) $K_2(\bar{x}, \bar{y}) > 0$ *for all* $\bar{x}, \bar{y} \in \Delta$.

Then property (P3) *holds for L.*

Remark. It is also true that the conclusion of Theorem 4 holds if (ii) is replaced by the weaker condition

$$\text{(ii')} \begin{cases} K_2(\bar{x}, \bar{y}) \geq 0 & \text{for } \bar{x}, \bar{y} \in \Delta \\ K_2(\bar{x}, \bar{y}) > 0 & \text{if } \bar{x}, \bar{y} \in \Delta,\ x_2 > y_1, \text{ and } x_1 < y_2. \end{cases}$$

This fact would be useful in applying Theorem 4 to the Benjamin–Ono solitary wave, say, for which (ii') holds but (ii) does not.

Proof. It is required to show that the equation

$$Mh + (C - 1)h - f'(\varphi)h = 0 \tag{4.1}$$

has a one-dimensional solution space in L_2. After application of the Fourier transform and division by the function $m(x) = \alpha(x) + (C - 1)$ (where x now represents the variable in Fourier transform space), equation (4.1) may be rewritten in the form

$$\int_{-\infty}^{\infty} G(x, y)g(y)\,dy = g(x),$$

where $g(x)=\hat{h}(x)$ and $G(x, y)=K(x-y)/m(x)$. Therefore our goal may be restated as showing that $\lambda=1$ is a simple eigenvalue for the operator S acting on L_2 defined by

$$Sg(x)=\int_{-\infty}^{\infty} G(x, y)g(y)\,dy.$$

To study the spectrum of S, use will be made of the theory of totally positive operators which was developed for precisely this sort of purpose by Gantmacher & Krein (1960) and Karlin (1964). Of crucial importance will be the operator S_2 defined on $L_2(\Delta)$ by

$$S_2g(\bar{x})=\iint_{\Delta} G_2(\bar{x}, \bar{y})g(\bar{y})\,d\bar{y},$$

where

$$G_2(\bar{x}, \bar{y})=\frac{K_2(\bar{x}, \bar{y})}{m(x_1)m(x_2)},$$

and K_2 is as defined earlier. The key identity relating the operators S and S_2 is stated in the following lemma, whose proof is elementary.

LEMMA 5 *For any two functions $f_1, f_2 \in L_2(\mathbb{R})$, define $f_1 \wedge f_2$ on $L_2(\Delta)$ by*

$$(f_1 \wedge f_2)(x_1, x_2)=f_1(x_1)f_2(x_2)-f_1(x_2)f_2(x_1).$$

Then

$$S_2(f_1 \wedge f_2)=Sf_1 \wedge Sf_2.$$

The spectral analysis of S and S_2 necessary to complete the proof of Theorem 4 is presented in the next four lemmas. Define the Hilbert space $\mathscr{W}$ to be the set of all functions $g \in L_2(\mathbb{R})$ for which

$$\|g\|_{\mathscr{W}}=\left(\int_{-\infty}^{\infty}|g(x)|^2m(x)\,dx\right)^{\frac{1}{2}}<\infty$$

with the inner product

$$(g, h)_{\mathscr{W}}=\int_{-\infty}^{\infty} g(x)h(x)m(x)\,dx,$$

and let $\mathscr{X}$ be the corresponding Hilbert space of functions $g \in L_2(\Delta)$ such that

$$\|g\|_{\mathscr{X}}=\left(\iint_{\Delta}|g(\bar{x})|^2m(x_1)m(x_2)\,d\bar{x}\right)^{\frac{1}{2}}<\infty$$

with the inner product

$$(g, h)_{\mathscr{X}}=\iint_{\Delta} g(\bar{x})h(\bar{x})m(x_1)m(x_2)\,d\bar{x}.$$

LEMMA 6 *Suppose $g \in L_2(\mathbb{R})$ is an eigenfunction of S for a nonzero eigenvalue γ. Then the function g lies in $\mathscr{W}$ and $g(x)$ is a continuous function of $x \in \mathbb{R}$.*

Proof. It will be shown that Sg is continuous and that $S^2g = S(Sg) \in \mathcal{W}$; since $g = (1/\gamma)Sg = (1/\gamma^2)S^2g$, this will suffice to prove the lemma. For brevity of notation, let $\mu(x) = 1/m(x)$. One then has

$$|Sg(x) - Sg(x')| \leqslant \int_{-\infty}^{\infty} |G(x, y) - G(x', y)|\, |g(y)|\, \mathrm{d}y$$
$$\leqslant \left(\int_{-\infty}^{\infty} |K(x-y)\mu(x) - K(x'-y)\mu(x')|^2\, \mathrm{d}y\right)^{\frac{1}{2}} |g|_2$$
$$\leqslant \left[\mu(x)\left(\int_{-\infty}^{\infty} |K(x-y) - K(x'-y)|^2\, \mathrm{d}y\right)^{\frac{1}{2}} + |\mu(x) - \mu(x')|\, |K|_2\right] |g|_2,$$

and the expression on the right-hand side of this inequality tends to zero as $x \to x'$, thus demonstrating that $Sg(x)$ is continuous.

The proof that $\|S^2g\|_{\mathcal{W}}$ is finite, although somewhat complicated by the relatively weak decay to zero of $\mu(x)$ as $|x| \to \infty$, nevertheless proceeds by straightforward applications of Minkowski's integral inequality, Hölder's inequality, and Young's convolution inequality. The following sequence of estimates establishes the desired result.

$$\|S^2g\|_{\mathcal{W}} = \left\|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} K(\bullet - z)\mu(\bullet)K(z-w)\mu(z)g(w)\, \mathrm{d}w\, \mathrm{d}z\right\|_{\mathcal{W}}$$
$$\leqslant \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \|K(\bullet - z)\mu(\bullet)\|_{\mathcal{W}} K(z-w)\mu(z)\, |g(w)|\, \mathrm{d}w\, \mathrm{d}z$$
$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} [K(y-z)]^2\mu(y)\, \mathrm{d}y\right)^{\frac{1}{2}} K(z-w)\mu(z)\, |g(w)|\, \mathrm{d}w\, \mathrm{d}z$$
$$= \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} (K^2 * \mu)^{\frac{1}{2}}(z)\mu(z)^{\frac{3}{8}}K(z-w)\mu(z)^{\frac{5}{8}}\, \mathrm{d}z\right) |g(w)|\, \mathrm{d}_w$$
$$\leqslant \int_{-\infty}^{\infty} (|K^2 * \mu|_2^{\frac{1}{2}}\, |\mu^{\frac{3}{2}}|_1^{\frac{1}{4}}[(K^2 * \mu^{\frac{5}{4}})(w)]^{\frac{1}{2}})\, |g(w)|\, \mathrm{d}w$$
$$\leqslant |K^2 * \mu|_2^{\frac{1}{2}}\, |\mu^{\frac{3}{2}}|_1^{\frac{1}{4}}\, |K^2 * \mu^{\frac{5}{4}}|_1^{\frac{1}{2}}\, |g|_2$$
$$\leqslant |K^2|_1 |\mu|_2^{\frac{1}{2}} |\mu^{\frac{3}{2}}|_1^{\frac{1}{4}}\, |\mu^{\frac{5}{4}}|_1^{\frac{1}{2}}\, |g|_2.$$

Since $|\mu(y)| \leqslant b(1 + |y|)^{-1}$ for some $b > 0$, all the quantities in the final expression of the preceding inequality are finite. □

LEMMA 7 *The restriction of S to $\mathcal{W}$ is a compact self-adjoint operator on $\mathcal{W}$ and the restriction of S_2 to $\mathcal{X}$ is a compact self-adjoint operator on $\mathcal{X}$.*

Proof. If $\mathcal{W}$ is viewed as the weighted L_2 space $L_2(\mathrm{d}\nu(x))$, where $\mathrm{d}\nu(x)$ is the positive measure on $\mathbb{R}$ given by $\mathrm{d}\nu(x) = m(x)\, \mathrm{d}x$, then the action of S on an element g of $\mathcal{W}$ may be expressed as

$$Sg(x) = \int_{-\infty}^{\infty} \frac{K(x-y)}{m(x)} g(y)\, \mathrm{d}y = \int_{-\infty}^{\infty} \tilde{G}(x, y)g(y)\, \mathrm{d}\nu(y),$$

where $\tilde{G}(x, y) = K(x-y)/[m(x)m(y)]$ satisfies $\tilde{G}(x, y) = \tilde{G}(y, x)$. The assertions of the lemma concerning S will then follow immediately from the theory of Hilbert–Schmidt operators once it is shown that $\tilde{G}(x, y) \in L_2(dv \times dv)$, a fact that follows at once from the inequality

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\tilde{G}(x, y)]^2 \, dv(x)\, dv(y) = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} K^2(x-y)\mu(y)\, dy\right)\mu(x)\, dx$$

$$= \int_{-\infty}^{\infty} (K^2 * \mu)(x)\mu(x)\, dx \leq |K^2 * \mu|_2 |\mu|_2$$

$$\leq |K^2|_1 \, |\mu|_2^2 < \infty.$$

A similar estimate shows S_2 to be a Hilbert–Schmidt operator on $\mathscr{X}$. The lemma is thus established. □

A consequence of Lemma 6 is that, to prove Theorem 4, it suffices to show that $\lambda = 1$ is a simple eigenvalue for S viewed as an operator on $\mathscr{W}$. Therefore, in the spectral analysis of S that follows, the space upon which S will be considered to act will always be $\mathscr{W}$. By Lemma 7 and the spectral theorem for compact self-adjoint operators on a Hilbert space, $\mathscr{W}$ has an orthonormal basis $\{\psi_i\}_{i=0}^{\infty}$ consisting of eigenvectors for S which correspond to real eigenvalues $\{\lambda_i\}_{i=0}^{\infty}$ whose only possible accumulation point is zero. We number the eigenvalues of S (which are not necessarily distinct) in such a way that $|\lambda_0| \geq |\lambda_1| \geq |\lambda_2| \geq \cdots \geq 0$. Similarly, we denote the eigenvalues of S_2 acting on $\mathscr{X}$ by $\{\mu_i\}_{i=0}^{\infty}$, where $|\mu_0| \geq |\mu_1| \geq \cdots \geq 0$.

The next two lemmas form the backbone of the theory of totally positive operators. Their statement and proof appear in Karlin (1964) in a somewhat different context. The proofs provided here are for the reader's convenience, and take account of the simplification available due to the self-adjointness of the operators in question.

LEMMA 8 (a) *The eigenvalue λ_0 of S is positive, simple, and has a strictly positive eigenfunction $\psi_0(x)$. Furthermore, we have $|\lambda_1| < \lambda_0$. Let $\mathscr{M}_0$ be the one-dimensional subspace of $\mathscr{W}$ spanned by ψ_0. Then there exists a projection P of $\mathscr{W}$ onto $\mathscr{M}_0$ such that $S = \lambda_0 P + Q$, where the spectral radius of Q is $|\lambda_1|$. In addition, one has $\lambda_0^{-n} S^n \to P$ as $n \to \infty$, in the strong operator topology.*

(b) *The statements in part* (a) *are all valid if S is replaced by S_2, the eigenvalues λ_0 and λ_1 are replaced by μ_0 and μ_1, and P, $\mathscr{M}_0$, and Q are replaced by the appropriate operators and subspaces of $\mathscr{X}$.*

Proof. The eigenvalue λ_0 of S having the largest absolute value is determined by the elementary formula

$$\lambda_0 = \pm \sup_{\|g\|_{\mathscr{W}}=1} |\langle Sg, g\rangle_{\mathscr{W}}|.$$

Let ψ be any eigenfunction of S corresponding to the eigenvalue λ_0 and suppose $\|\psi\|_{\mathscr{W}} = 1$. By Lemma 7, ψ is continuous, and so, because the kernel G of S is everywhere positive, if ψ takes both positive and negative values, then $S(|\psi|) >$

$|S(\psi)| = |\lambda_0|\,|\psi|$. But then $\langle S(|\psi|), |\psi|\rangle > |\lambda_0|$, a contradiction. Thus, we may take it that any eigenfunction ψ associated to the eigenvalue λ_0 is one-signed, and in particular that there is an eigenfunction ψ_0 which is nonnegative. But since $\lambda_0\psi_0 = S(\psi_0)$, it follows first that $\lambda_0 > 0$, and then that $\psi_0(x) > 0$ for all $x \in \mathbb{R}$. Such a ψ_0 cannot be orthogonal to any nontrivial one-signed function in $\mathcal{W}$, and so λ_0 is a simple eigenvalue. Notice that the preceding argument shows also that $-\lambda_0$ cannot be an eigenvalue of S, and it therefore follows that $|\lambda_1| < \lambda_0$.

The decomposition of S into the form $S = \lambda_0 P + Q$, where P is the orthogonal projection on $\mathcal{M}_0$, $PQ = QP = 0$, and spectrum (Q) = spectrum $(P) - \{\lambda_0\}$, is a standard result from the spectral theory of self-adjoint operators (see Kato, 1984: Chap. V.3.5). In the present case, the spectral radius of Q, being the supremum of the absolute values of the numbers in the spectrum of Q, is $|\lambda_1|$, which is strictly less than λ_0. In consequence of this observation, it follows that

$$\lim_{n\to\infty} \left|\left|\left| \frac{S^n}{\lambda_0^n} - P \right|\right|\right| = \lim_{n\to\infty} \left|\left|\left| \frac{(\lambda_0 P)^n + Q^n}{\lambda_0^n} - P \right|\right|\right| = \lim_{n\to\infty} \left|\left|\left| \frac{Q^n}{\lambda_0^n} \right|\right|\right| \leqslant \lim_{n\to\infty} \frac{|\lambda_1|^n}{\lambda_0^n} = 0,$$

where triple bars have been used to represent the strong norm on the space of bounded operators on $\mathcal{W}$. This completes the proof of the assertions in part (a) of the lemma. Since the arguments given in the above proof clearly apply just as well to S_2 as to S, there is no need to say more concerning part (b). □

LEMMA 9 *In the notation introduced prior to the statement of Lemma* 8, *the identity* $\mu_0 = \lambda_0\lambda_1$ *holds.*

Proof. Notice that $\lambda_0\lambda_1$ is an eigenvalue of S_2 with eigenfunction $\psi_0 \wedge \psi_1$. Hence, by Lemma 8(b), to prove Lemma 9 it suffices to show that $|\lambda_0\lambda_1| \geqslant \mu_0$. Suppose to the contrary that $|\lambda_1| < (\mu_0/\lambda_0)$. Let P be as in Lemma 8 and write $\mathcal{W} = \mathcal{M}_0 \oplus \mathcal{N}$, where $\mathcal{M}_0$ = range P = span $\{\psi_0\}$ and $\mathcal{N}$ = ker P. Notice that the restriction of S to $\mathcal{N}$ has spectral radius $|\lambda_1|$. Let f_1 and f_2 be arbitrary elements of $\mathcal{W}$, and write $f_1 = \eta_1\psi_0 + \omega_1$, $f_2 = \eta_2\psi_0 + \omega_2$, where $\eta_1, \eta_2 \in \mathbb{R}$ and $\omega_1, \omega_2 \in \mathcal{N}$. Then we find that, for any positive integer n,

$$\begin{aligned}(S_2/\mu_0)^n(f_1 \wedge f_2)(x_1, x_2) = {} & \eta_1[\varphi_0(x_1)(S/\beta)^n\psi_2(x_2) - \varphi_0(x_2)(S/\beta)^n\psi_2(x_1)] \\ & + \eta_2[\varphi_0(x_2)(S/\beta)^n\psi_1(x_1) - \varphi_0(x_1)(S/\beta)^n\psi_1(x_2)] \\ & + (S/\lambda_0)^n\psi_1(x_1)(S/\beta)^n\psi_2(x_2) \\ & - (S/\lambda_0)^n\psi_1(x_2)(S/\beta)^n\psi_2(x_1),\end{aligned}$$

where $\beta = (\mu_0/\lambda_0) > |\lambda_1|$. Since $(S/\lambda_0)^n$ converges strongly as $n \to \infty$, and β is strictly greater than the spectral radius of the restriction of S to $\mathcal{N}$, each term on the right-hand side of the preceding equality tends to zero as $n \to \infty$. But the set of all $f_1 \wedge f_2$ with $f_1, f_2 \in \mathcal{W}$ is dense in $\mathcal{X}$, and so the preceding computation shows that $(S_2/\mu_0)^n g \to 0$ as $n \to \infty$ for any $g \in \mathcal{X}$. This then contradicts Lemma 8(b), which asserts that $(S_2/\mu_0)^n$ converges strongly to a nonzero projection operator. The proof of the lemma is therefore complete. □

The next step in the proof of Theorem 4 is to show that the eigenvalue λ_1 of S is simple. Let ψ_1 be an eigenfunction for λ_1, and write $\psi_1 = \psi_1^e + \psi_1^o$, where ψ_1^e

and ψ_1^o are the even and odd parts of ψ_1 (that is, $\psi_1^e(x) = \frac{1}{2}[\psi_1(x) + \psi_1(-x)]$ and $\psi_1^o(x) = \frac{1}{2}[\psi_1(x) - \psi_1(-x)]$). Both ψ_1^e and ψ_1^o are also eigenfunctions for λ_1, as follows from the fact that S maps even functions to even functions and odd functions to odd functions in $\mathcal{W}$. Furthermore, $\psi_1^e \wedge \psi_0$ is an eigenfunction of S_2 with eigenvalue $\lambda_1\lambda_0 = \mu_0$. Hence, by Lemma 8(b), $\psi_1^e \wedge \psi_0$ is either identically zero on Δ or does not vanish at all on Δ. Since $(\psi_1^e \wedge \psi_0)(x_1, x_2) = \psi_1^e(x_1)\psi_0(x_2) - \psi_1^e(x_2)\psi_0(x_1)$ for $x_1 < x_2$, it follows that, if $\psi_1^e \not\equiv 0$ on $\mathbb{R}$, then ψ_1^e can have at most one zero on $\mathbb{R}$. Then, since ψ_1^e is even and continuous (by Lemma 6), this implies that either $\psi_1^e \equiv 0$ or else ψ_1^e does not vanish on $\mathbb{R}$ except possibly at zero, and in any case it would then be of one sign everywhere else. But the latter possibility is excluded by the fact that ψ_1^e and ψ_0, being eigenfunctions of S for distinct eigenvalues, must satisfy $(\psi_1^e, \psi_0)_{\mathcal{W}} = 0$. Therefore, $\psi_1^e \equiv 0$, and so ψ_1 is odd. Again, applying Lemma 8(b) to the eigenfunction $\psi_1 \wedge \psi_0$ for the eigenvalue $\mu_0 = \lambda_0\lambda_1$, one concludes that ψ_1 can have at most one zero, and since ψ_1 is odd, this zero must be at the origin.

The function ψ_1 in the preceding paragraph was an arbitrary eigenfunction for λ_1, and consequently what has actually been shown is that any eigenfunction for λ_1 must be an odd function which vanishes only at $x = 0$. But, clearly, no two such functions can be orthogonal in $\mathcal{W}$. Therefore λ_1 must be a simple eigenvalue of S.

To complete the proof of Theorem 4, then, it remains only to show that $\lambda_1 = 1$. Now equation (3.1) shows that φ' is an eigenfunction of L for the eigenvalue 0, and so φ' is an eigenfunction of S for the eigenvalue 1. But $\hat{\varphi}$ is, by assumption, a positive even function on $\mathbb{R}$, and therefore $\varphi'(x) = -ix\hat{\varphi}(x)$ is an odd function on $\mathbb{R}$ which vanishes only at $x = 0$. Since ψ_1 is also odd and vanishes only at $x = 0$, we must have $(\psi_1, \varphi')_{\mathcal{W}} \neq 0$. It follows that ψ_1 and φ' cannot be eigenfunctions of S for distinct eigenvalues. This proves that λ_1 must equal 1. □

The final result of this section is a lemma which is useful for verifying condition (ii) of the preceding theorem. It is an instance of more general results detailed by Karlin (1968).

LEMMA 10 *Suppose K is twice differentiable on $\mathbb{R}$ and satisfies $K(x) > 0$ for $x \in \mathbb{R}$ and $(d^2/dx^2)\,[\log K(x)] < 0$ for $x \neq 0$. Then $K_2(\bar{x}, \bar{y}) > 0$ for all $\bar{x},\bar{y} \in \Delta$.*

Proof. The hypotheses imply that $K'(x)/K(x)$ is strictly decreasing on $\mathbb{R}$. Supposing $\bar{x} = (x_1, x_2)$ and $\bar{y} = (y_1, y_2)$ are in Δ, then, for any $x \in \mathbb{R}$, we have

$$\frac{K'(x-y_1)}{K(x-y_1)} < \frac{K'(x-y_2)}{K(x-y_2)}. \tag{4.2}$$

The mean-value theorem applied with respect to the variable x allows us to conclude that

$$\begin{aligned} K(x_1-y_1)K(x_2-y_2) - K(x_1-y_2)K(x_2-y_1) &= K(x_1-y_1)K(x_2-y_1)\left(\frac{K(x_2-y_2)}{K(x_2-y_1)} - \frac{K(x_1-y_2)}{K(x_1-y_1)}\right) \\ &= K(x_1-y_1)K(x_1-y_2)(x_2-x_1)\frac{d}{dx}\left(\frac{K(x-y_2)}{K(x-y_1)}\right)\bigg|_{x=\eta} \end{aligned}$$

$$= K(x_1 - y_1)K(x_1 - y_2)(x_2 - x_1)\left\{\frac{K(\eta - y_1)K'(\eta - y_2) - K(\eta - y_2)K'(\eta - y_1)}{[K(\eta - y_1)]^2}\right\}$$

for some η with $x_1 < \eta < x_2$. But the term in braces is positive by (4.2), and so the lemma is established. □

5. The ILW equation

The theory developed in the preceding two sections will now be applied to the ILW equation

$$u_t + u_x + uu_x - (Mu)_x = 0, \tag{5.1}$$

where $f(u) = \frac{1}{2}u^2$ and M is the Fourier multiplier operator with symbol

$$\alpha(k) = \alpha_H(k) = k \coth kH - \frac{1}{H},$$

with $H > 0$ a fixed constant. Equation (5.1) is derived, for example, by Kubota *et al.* (1978) as a model equation for long weakly nonlinear internal gravity waves in a stratified fluid of finite depth. The parameter H, which is related to the depth of the fluid, is allowed to take any positive value, so that (5.1) actually represents a one-parameter family of equations.

Joseph (1977) found that, for any $C > 1$ and $H > 0$, equation (5.1) has the solitary-wave solution $u(x, t) = \varphi(x - Ct)$, where φ is given by

$$\varphi(y) = \frac{b}{\cosh^2 ay + (b^2/16a^2)\sinh^2 ay}. \tag{5.2}$$

Here $a \in (0, \pi/2H)$ and $b \in (0, \infty)$ are determined uniquely in terms of C and H by the equations

$$1 - \frac{aH}{\tan aH} + aH \tan aH = (C - 1)H, \qquad aH \tan aH = \tfrac{1}{4}bH. \tag{5.3}$$

The goal of this section is a proof of the stability of Joseph's solitary waves for all values of $C > 1$ and $H > 0$. Of the four conditions which must be verified in order to prove stability using Theorem 2, (P1) and (P3) yield easily to the techniques of Section 4, while (P4) is verified by an elementary computation. The remaining condition (P2) will be handled by perturbation theory as in Albert *et al.* (1987), together with a continuity argument.

It is appropriate here to mention an alternative method, due to Weinstein (1987), for verifying that solitary waves satisfy condition (P2). Weinstein shows that (P2) holds for those solitary-wave solutions of (2.1) which are minimizers of a certain nonlinear functional on $\mathscr{X}$. Unfortunately, there is a technical difficulty involved in applying this elegant approach to the problem under discussion here. Briefly put, the difficulty arises from the fact that it has not yet been proved that the solitary-wave solutions of ILW which minimize Weinstein's functional are the same as those given by Joseph's explicit formulae (5.2) and (5.3), although the available evidence certainly suggests that this is in fact the case. While the technique used below for verifying (P2) is special to the ILW equation, it does have the advantage of circumventing this difficulty.

THEOREM 11 *For any $C>1$ and $H>0$, the solitary wave defined in* (5.2) & (5.3) is stable as defined before the statement of Theorem 2.

Remark. The well-posedness properties of ILW are studied in Abdelouhab *et al.* (1989), where it is shown that the quantity T^* defined in Theorem 1 is equal to infinity for all initial data that lies in H^s for $s>\frac{3}{2}$.

Proof. It suffices to show that conditions (P1)–(P4) of Theorem 2 are satisfied for the operator L associated with the solitary wave (5.2). As was observed in Albert *et al.* (1987), a change of variables shows that (P1)–(P4) hold for L with $b=b_0$ and $H=H_0$ if and only if they hold with $b=b_1$ and $H=H_1$ whenever $b_0H_0=b_1H_1$. Therefore, it suffices to verify (P1)–(P4) for a fixed value of b, with H allowed to take arbitrary positive values. For convenience, choose $b=4$, in which case (5.2) may be rewritten as

$$\varphi(y)=\frac{8a^2}{a^2+1}\left(\frac{1}{\cosh 2ay+\cos\delta}\right), \tag{5.4}$$

where

$$\cos\delta=\frac{a^2-1}{a^2+1}$$

and δ lies in the range $(0,\pi)$.

Property (P1) is verified for ILW solitary waves by application of Theorem 4. The inequality $K_\mu(x)>0$ for all $\mu>0$ and $x\in\mathbb{R}$ was shown to be valid for the ILW equation in Albert *et al.* (1987: p. 364), using a residue calculation.

To prove property (P3) holds, one uses Theorem 4. From (5.4) and a table of Fourier transforms, it is found that

$$K(x)=\hat{\varphi}(x)=\left(\frac{4a}{a^2+1}\right)\left(\frac{1}{\sin\delta}\right)\frac{\sinh(\delta x/2a)}{\sinh(\pi x/2a)}.$$

Thus $K(x)>0$ for $x\in\mathbb{R}$, and it is readily verified that $(d^2/dx^2)[\log K(x)]<0$ for $x\neq 0$. The validity of (P3) then follows from Theorem 4 and Lemma 10.

To verify property (P2), let $\mathscr{C}$ be the set given by

$$\mathscr{C}=\{\omega>0:\text{(P2) is false for}\quad H=\omega \text{ and } b=4\}.$$

It is intended to show that the assumption $\mathscr{C}\neq\varnothing$ leads to a contradiction.

Assume $\mathscr{C}\neq\varnothing$, so that $\omega_0=\inf\mathscr{C}$ exists. As shown in Albert *et al.* (1987) by use of the perturbation theory of operators, (P2) holds for all sufficiently small values of the quantity bH. Therefore $\omega_0>0$. For any $\omega>0$, let φ_ω be the solitary wave given by (5.4) & (5.3) with $b=4$ and $H=\omega$, and let L_ω denote the operator associated with φ_ω as in (3.2). By definition of ω_0, there exists a sequence $\{\omega_i\}_{i=1}^\infty$ converging to ω_0 from above and eigenvalues α_i of L_{ω_i} such that $\alpha_i\in(\lambda_i,0)$, where λ_i is the least eigenvalue of L_{ω_i}.

As explained in Albert *et al.* (1987), the spectrum $\sigma(L_\omega)$ of L_ω depends continuously on ω, in the following sense. Fix $\omega = \bar{\omega}$ and let Γ be any contour in the complex plane, disjoint from $\sigma(L_{\bar{\omega}})$ and surrounding a set of eigenvalues of $L_{\bar{\omega}}$ with total multiplicity m. Then, for ω sufficiently near to $\bar{\omega}$, the contour Γ also surrounds a set of eigenvalues of L_ω with total multiplicity m.

In particular, if λ_0 is the least eigenvalue of L_{ω_0}, then $\lambda_i \to \lambda_0$ as $i \to \infty$. Therefore, by passing to a subsequence if necessary, one may assume that there exists $\alpha_0 \in [\lambda_0, 0]$ such that $\alpha_i \to \alpha_0$.

It will now be argued that each of the three possibilities $\alpha_0 = \lambda_0$, $\alpha_0 = 0$, and $\alpha_0 \in (\lambda_0, 0)$ leads to a contradiction. If $\alpha_0 = \lambda_0$, draw a contour Γ in $\mathbb{C}$ such that λ_0 is the only eigenvalue of L_{ω_0} enclosed by Γ. Since λ_0 is simple, it follows that when i is large, L_{ω_i} has only one (simple) eigenvalue enclosed by Γ. But this contradicts the fact that both λ_i and α_i are enclosed by Γ for large values of i. For $\alpha_0 = 0$, a contradiction may be reached by the same argument. (Notice that, at this stage of the proof, crucial use is made of the fact, established above, that (P3) holds for L_ω for all $\omega > 0$.)

Finally, if $\lambda_0 < \alpha_0 < 0$, draw a contour Γ_1 enclosing only the eigenvalue λ_0 of L_{ω_0} and a contour Γ_2 enclosing only the eigenvalue α_0 of L_{ω_0} so that Γ_1 and Γ_2 have disjoint interiors and do not intersect the nonnegative real axis. Since $\sigma(L_\omega)$ varies continuously, there exists a value of ω less than ω_0 for which Γ_1 and Γ_2 each enclose a (negative) eigenvalue of L_ω. But then $\omega \in \mathscr{E}$, contradicting the definition of ω_0.

These contradictions show that $\mathscr{E} = \emptyset$ necessarily, and hence (P2) is seen to be true for $b = 4$ and all $H > 0$.

To complete the proof of the theorem, it remains only to show that (P4) holds for all $C > 1$ and $H > 0$. As remarked above, it is sufficient to show that (P4) holds for a fixed value of H and arbitrary values of $C > 1$.

Setting $H = 1$, one sees from (5.3) that a ranges from 0 to $\frac{1}{2}\pi$ as C ranges from 1 to ∞, and that

$$\frac{\mathrm{d}C}{\mathrm{d}a} = \frac{4a - \sin 4a}{(\sin 2a)^2} > 0$$

for all $a > 0$. By the chain rule,

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(\int_{-\infty}^{\infty} \varphi_C^2(x)\,\mathrm{d}x\right) = \frac{\mathrm{d}}{\mathrm{d}C}\left(\int_{-\infty}^{\infty} \varphi_C^2(x)\,\mathrm{d}x\right)\frac{\mathrm{d}C}{\mathrm{d}a},$$

and thus it suffices to check that the left-hand side of the last equation is positive for all $a \in (0, \frac{1}{2}\pi)$.

From (5.3), we have $b = 4a \tan a$, and so (5.2) gives

$$\varphi_C(x) = \frac{4\,a \tan a}{\cosh^2 ax + \tan^2 a \sinh^2 ax}.$$

An explicit integration then shows that

$$\frac{\mathrm{d}}{\mathrm{d}a}\left(\int_{-\infty}^{\infty} \varphi_C^2(x)\,\mathrm{d}x\right) = \frac{\mathrm{d}}{\mathrm{d}a}\left[16a\left(1 - \frac{2a}{\tan 2a}\right)\right] = \frac{16}{\sin^2 a}\left(\sin^2 2a - 2a \sin 4a + 4a^2\right),$$

which is positive for all $a > 0$. Thus (P4) is verified, and the proof of Theorem 11 is complete. □

Acknowledgements

The research leading to this paper was partially supported by the National Science Foundation. The work was completed while both authors were in residence at the Institute for Mathematics and its Applications at the University of Minnesota. The authors would like to record their debt to two anonymous referees, whose remarks were quite helpful.

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Appendix

In this Appendix, we return to Theorem 2 and its proof. Proofs of the assertions in Theorem 2 (and in the accompanying Remark) may be found in the papers of Weinstein (1983, 1985, 1986). A concise proof using a somewhat different approach may be found in Section 5 of Bona *et al.* (1987). However, there is a step missing in the exposition contained therein, and the opportunity is taken here to repair this omission.

In Lemma 5.1 of Bona *et al.* (1987), it is shown that, if $y \in \mathscr{X}$ is nonzero and satisfies $(y, \varphi) = (y, \varphi') = 0$, then (under the assumptions of Theorem 2 above) one has $\langle Ly, y \rangle > 0$ where $\langle V, f \rangle$ denotes the $\mathscr{X}^* - \mathscr{X}$ pairing mentioned earlier. However, the proof of Theorem 2 given in the last-quoted reference actually requires a somewhat stronger result, which is stated and proved now.

LEMMA *Suppose properties* (P1)–(P4) (*or* (P4′)) *hold for* φ. *Let* η *be defined by*

$$\eta = \inf \{ \langle Ly, y \rangle : y \in X, (y, \varphi) = (y, \varphi') = 0, \textit{ and } (y, y) = 1 \}.$$

Then the constant η *is strictly positive.*

Proof. In light of Lemma 5.1 of Bona *et al.* (1987), it suffices to prove that $\eta \neq 0$. To accomplish this, it will be supposed that $\eta = 0$, with a view to obtaining a contradiction.

Let $\{g_n\}_{n=1}^{\infty}$ be a sequence of functions in $\mathscr{X}$ such that $(g_n, \varphi) = (g_n, \varphi') = 0$ and $(g_n, g_n) = 1$, for all n, and $\lim_{n\to\infty} \langle Lg_n, g_n \rangle = 0$. It follows from the estimate

$$\begin{aligned} 0 \leq \langle Mg_n, g_n \rangle &= \langle Lg_n, g_n \rangle - (C_0 - 1)(g_n, g_n) + \int_{-\infty}^{\infty} f'(\varphi(x))[g_n(x)]^2 \, dx \\ &\leq \langle Lg_n, g_n \rangle + \sup_{0 \leq s \leq |\varphi|_\infty} |f'(s)| \end{aligned}$$

that the sequence $\{\langle Mg_n, g_n \rangle\}_{n=1}^{\infty}$ is bounded as n ranges over the positive integers. Hence $\{g_n\}_{n=1}^{\infty}$ is bounded in $\mathscr{X}$, and so some subsequence of $\{g_n\}_{n=1}^{\infty}$ must converge weakly in $\mathscr{X}$ to a function g_*. We will continue to write $\{g_n\}_{n=1}^{\infty}$ even though reference will henceforth be to the subsequence. In fact, in the argument to follow, several further subsequences will be extracted, all of which will be written as simply $\{g_n\}_{n=1}^{\infty}$ rather than with some more definite relabelling of the indices.

Because of condition (2.2) on the symbol α of the operator M, the space $\mathscr{X}$ is continuously embedded in $H^{\frac{1}{2}}$, and hence in L_p for all finite $p \geq 2$. It follows that $\{g_n\}_{n=1}^{\infty}$ converges weakly to g_* in both L_2 and L_4, and in particular, $\{g_n\}_{n=1}^{\infty} \cup \{g_*\}$ is a bounded set in L_4. Since $H^{\frac{1}{2}}((-k, k))$ is compactly imbedded in $L_2((-k, k))$ for any finite, positive value of k, and since a sequence that is strongly convergent to a function in $L_2((-k, k))$ has a subsequence that is convergent almost everywhere to that function, a Cantor diagonalization argument leads to a subsequence $\{g_n\}_{n=1}^{\infty}$ that converges to g_* in $L_{2,\mathrm{loc}}(\mathbb{R})$ and pointwise almost everywhere in $\mathbb{R}$. In consequence of the fact that $\{g_n\}_{n=1}^{\infty}$ is bounded in L_4, it follows that $\{g_n^2\}_{n=1}^{\infty}$ is bounded in L_2, and hence passing to a further subsequence allows us to assume that $\{g_n^2\}_{n=1}^{\infty}$ converges to some L_2 function G, say, weakly in L_2. It follows that $\{g_n^2\}_{n=1}^{\infty}$ converges in the sense of

Schwartz distributions $\mathfrak{D}'(\mathbb{R})$ to G. But, if $\psi \in C_0^\infty(\mathbb{R})$ with support contained in the interval $[a, b]$, say, then Hölder's inequality implies that

$$\left|\int_{-\infty}^{\infty} [g_n^2(x) - g_*^2(x)]\psi(x)\,dx\right| \leq \|g_n - g_*\|_{L_2([a,b])}\, \|\psi\|_{L_4}(\|g_n\|_{L_4} + \|g_*\|_{L_4}).$$

The second factor on the right-hand side is a constant, the third factor is bounded independently of n, and the first factor tends to zero as n tends to infinity. It is therefore concluded that $g_n^2 \to g_*^2$ as $n \to \infty$ in $\mathfrak{D}'(\mathbb{R})$; whence $G = g_*^2$. Thus, by successive extraction of subsequences, we are left with a subsequence $\{g_k\}_{k=1}^\infty$ of the original sequence $\{g_n\}_{n=1}^\infty$ for which

$$0 = \lim_{k\to\infty} (\varphi, g_k) = (\varphi, g_*),$$

$$0 = \lim_{k\to\infty} (\varphi', g_k) = (\varphi', g_*),$$

$$\lim_{k\to\infty} \big(f'(\varphi), g_k^2\big) = \big(f'(\varphi), g_*^2\big),$$

and, by the lower semicontinuity of the norm with respect to weak convergence,

$$\langle Mg_*, g_*\rangle \leq \liminf_{k\to\infty} \langle Mg_k, g_k\rangle$$

and

$$\|g_*\|_{L_2}^2 = (g_*, g_*) \leq \liminf_{k\to\infty} (g_k, g_k) = 1.$$

Combining the above relations, we have

$$\langle Lg_*, g_*\rangle \leq \liminf_{n\to\infty} \langle Lg_k, g_k\rangle = 0.$$

Furthermore, since $\langle Mg, g\rangle \geq 0$ for all $g \in \mathscr{X}$, it follows that

$$\begin{aligned} 0 &= \lim_{k\to\infty} \langle Lg_k, g_k\rangle \\ &\geq \liminf_{k\to\infty} \langle Mg_k, g_k\rangle + (C_0 - 1) - \int_{-\infty}^{\infty} f'\big(\varphi(x)\big) g_*^2(x)\,dx \\ &\geq (C_0 - 1) - \int_{-\infty}^{\infty} f'\big(\varphi(x)\big) g_*^2(x)\,dx. \end{aligned}$$

Since $C_0 - 1 > 0$, the last inequality shows that $g_* \neq 0$. Therefore, one may define $f_* = g_*/\|g_*\|_0$, thereby obtaining a function satisfying $(f_*, \varphi) = (f_*, \varphi') = 0$, $(f_*, f_*) = 1$, and

$$\langle Lf_*, f_*\rangle = \frac{1}{\|g_*\|_0^2}\langle Lg_*, g_*\rangle \leq 0.$$

But Lemma 5.1 of Bona *et al.* (1987) shows that such a function cannot exist. This contradiction means that η must be nonzero, and the proof of the lemma is now complete. □