On the well-posedness of the Cauchy problem for some nonlocal nonlinear Schrödinger equations

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Abstract

We prove well-posedness and ill-posedness results for two related nonlocal nonlinear Schrödinger equations on the line: the Gabitov-Turitsyn or dispersion-managed nonlinear Schrödinger equation, and a one-dimensional version of the continuous resonant equation derived by Faou, Germain, and Hani as the weakly nonlinear large box limit of the mass-critical nonlinear Schrödinger equation. It is shown that both equations are well-posed in Sobolev spaces H^r for all $r \geq 0$, and the latter equation is ill-posed in H^r for all $r < 0$, in the sense that solutions cannot depend uniformly continuously on the initial data.

1 Introduction

In this paper we prove well-posedness and ill-posedness results for the initial-value problem on the real line for the dispersion-managed nonlinear Schrödinger (DMNLS) equation

$$iu_t + \alpha u_{xx} + \int_0^1 g(s)\ T^{-1}(s) \left[|T(s)[u]|^2 T(s)[u] \right]\ ds = 0 \tag{1.1}$$

and the one-dimensional continuous resonant (1DCR) equation

$$iu_t + \int_{-\infty}^{\infty} U^{-1}(s) \left[|U(s)[u]|^4 U(s)[u] \right]\ ds = 0. \tag{1.2}$$

posed for complex-valued functions $u(x,t)$ defined for $x \in \mathbb{R}$ and $t \geq 0$. In equation (1.1), α is a real constant; $g : [0,1] \to \mathbf{R}$ is a bounded measurable function; and the operator $T(s)$ is the Fourier multiplier operator $T(s) = e^{iD(s)\partial_x^2}$, where $D : [0,1] \to \mathbf{R}$ is an absolutely continuous function. In equation (1.2), $U(s)$ is the Fourier multiplier $U(s) = e^{is\partial_x^2}$, defined for all $s \in \mathbf{R}$. (See (2.1) and (2.4) below for explicit definitions.)

The DMNLS equation (1.1) was derived by Gabitov and Turitsyn [GT] as a model for the propagation of weakly nonlinear, quasi-monochromatic electromagnetic pulses in an optical fiber whose dispersive properties alternate between strongly positive and strongly negative, and have mean value near zero. For a derivation of (1.1) from Maxwell's equations, the reader may consult chapter 10 of [A].

Equation (1.2) is a one-dimensional analogue of the continuous resonant equation, an equation which was derived in [FGH] as a model for the behavior of solutions of the cubic nonlinear Schrödinger equation on the two-dimensional torus in the weakly nonlinear, large-box limit. As is detailed in [FGH], the long-time asymptotics of small-amplitude solutions of the nonlinear Schrödinger equation on the torus is governed in certain parameter regimes by nonlinear resonances between their Fourier modes, as opposed to the situation in Euclidean space, where small-amplitude solutions scatter linearly. In fact, it is proved in [FGH] that the discrete Fourier modes of a small-amplitude solution of the cubic nonlinear Schrödinger equation on the two-dimensional torus $[0, L] \times [0, L]$ can be well approximated, in the limit as $L \to \infty$, by the Fourier transform of a solution of the continuous resonant equation on $\mathbf{R}^2$. At least formally, (1.2) can be derived as a one-dimensional analogue of the continuous resonant equation, governing the limiting behavior of periodic solutions of the one-dimensional quintic nonlinear Schrödinger equation

$$iu_t + \alpha u_{xx} + |u|^4 u = 0,$$

in the limit as the amplitude of the solutions tends to zero and the period of the solutions tends to infinity. (Note that the quintic nonlinear Schrödinger equation is mass-critical in one dimension, just as the cubic nonlinear Schrödinger equation is mass-critical in two dimensions.) Like the two-dimensional continuous resonant equation, (1.2) is of Hamiltonian form and appears to have a rich mathematical structure (see for instance the discussion of its standing-wave solutions in [HZ]).

In Theorems 2.3, 2.4, and 2.6 below, we prove that the DMNLS and 1DCR equations are globally well-posed in the L^2-based Sobolev spaces H^r for all $r \geq 0$, and that the 1DCR equation is ill-posed in H^r for all $r < 0$, at least in that the sense that a data-to-solution map cannot be defined on H^r which is uniformly continuous on every bounded set in H^r. The ill-posedness result is new. As for the well-posedness results, most of what they contain already appears in the literature, except that a proof of well-posedness for DMNLS in fractional Sobolev spaces H^r, $r > 0$, does not seem to have been given before for the general dispersion profiles $D(s)$ considered here. Still we thought it useful to include all the results in one unified exposition.

The results of this paper can be compared with existing well-posedness and ill-posedness results for the nonlinear Schrödinger equation

$$iu_t + \alpha u_{xx} + |u|^p u = 0, \tag{1.3}$$

with $\alpha \neq 0$. The natural choices of p to compare with (1.1) and (1.2) are $p = 2$ and $p = 4$. In the case when $p = 2$, equation (1.3) is also globally well-posed in H^r for all $r \geq 0$, in the sense that all the conclusions of Theorem 2.3 below hold for (1.3). In the mass-critical case when $p = 4$, local well-posedness holds in H^r for all $r \geq 0$, and solutions exist globally in L^2 for initial data with sufficiently small L^2 norm, but there exist solutions with initial data in H^1 which blow up in H^1 norm in finite time. (See chapters 4 and 6 of [C] for more details and further references). On the other hand, analogues of the ill-posedness result in Theorem 2.6 have been proved in spaces H^r for $-1/2 < r < 0$ for the case $p = 2$ in [KPV], and in L^2 for the case $p = 4$ in [BKPSV].

To prove our well-posedness result for the DMNLS equation, we show below in Lemma 3.7 that the nonlinear term $Q(u)$ in (1.1), defined by

$$Q(u) = \int_0^1 g(s)\ T^{-1}(s) \left[|T(s)[u]|^2 T(s)[u]\right]\ ds, \tag{1.4}$$

satisfies the estimate

$$\|Q(u)\|_{H^r} \le C\|u\|_{H^r}\|u\|_{L^2}^2 \tag{1.5}$$

for every $r \ge 0$, provided that $D(s)$ is absolutely continuous with derivative that is piecewise of one sign and bounded away from zero. Well-posedness then follows from the estimate (1.5) by standard arguments. We note that in the case $r = 0$ this estimate already appears in [HL], and for the case $r = 0$ and $D(s) = s$ was proved even earlier by Kunze in [Ku].

Note that whereas the NLS equation (1.3) is obviously ill-posed in the case when $\alpha = 0$, the DMNLS equation (1.1) is by contrast well-posed in this case. This is of particular interest in light of the remarkable fact that, when $\alpha = 0$, the DMNLS equation has stable solitary-wave solutions ([Ku], see also [KMZ] and [HL]). In fact, to complete the proof of stability, Hundertmark et al. have already observed in [HKS] that the DMNLS equation is globally well-posed in L^2 when $\alpha = 0$.

For the 1DCR equation (1.2), a similar estimate to (1.5) holds, from which well-posedness again follows by standard arguments. Indeed, for the two-dimensional version of this equation, global well-posedness in $H^r(\mathbf{R}^2)$ for all $r \ge 0$ was already proved by this procedure in [FGH].

To prove our ill-posedness result for the 1DCR equation, we follow the basic idea of the ill-posedness proofs in [KPV] and [BKPSV], which is to choose two different initial data from a certain family of special solutions of the equation which are close in H^r but which give rise to solutions whose velocities differ by a large amount, and which therefore separate in an arbitrarily small amount of time. Since special solutions of 1DCR with different translational velocities are not available, we use solutions with different phase velocities instead. On the other hand, since the 1DCR equation has a large collection of stationary-wave solutions to work with, we are able to obtain an ill-posedness result in H^r for every negative value of r; whereas the ill-posedness proofs for (1.3) in [KPV] and [BKPSV] required $r > -1/2$ for $p = 2$ and $r = 0$ for $p = 4$.

We remark that the technique used here and in [KPV] to obtain ill-posedness results does not apply to (1.1), because equation (1.1) does not enjoy the same dilation symmetry that equations (1.3) and (1.2) do. As far as we know, it remains an open question whether equation (1.1) is ill-posed in H^r for any $r < 0$.

Some of the well-posedness results proved here have appeared as part of the second author's Ph.D. thesis [K].

Notation. The set of natural numbers $\{1, 2, 3, \cdots\}$ and the set of all integers are written $\mathbf{N}$ and $\mathbf{Z}$, respectively. The set of all real numbers is denoted by $\mathbf{R}$.

For any measurable function f on $I \subseteq \mathbf{R}$ and any $p \in [1, \infty)$, we define

$$|f|_{L^p(I)} = \left(\int_I |f(x)|^p\ dx\right)^{\frac{1}{p}},$$

and $L^p(I)$ denotes the space of all f for which $\|f\|_{L^p(I)}$ is finite. The space $L^\infty(I)$ is defined as the space of all measurable functions f on I such that

$$|f|_\infty = \text{ess sup}_{x\in I}\,|f(x)|$$

is finite. In case $I = \mathbf{R}$, we use the abbreviation L^p for $L^p(\mathbf{R})$.

For $f, g \in L^2$ we define the L^2 inner product of f and g by

$$\langle f, g\rangle = \int_{-\infty}^{\infty} f(x)\overline{g(x)}\, dx,$$

where the bar denotes the complex conjugate.

If $f(x, y)$ is a measurable function defined for $(x, y) \in I \times J \subset \mathbf{R} \times \mathbf{R}$, and $1 \le p \le \infty$ and $1 \le q \le \infty$, we can, for each fixed y, view $f(x, y)$ as a function g_y of x. In other words, we define $g_y(x) = f(x, y)$. We then define h as a function of y by $h(y) = \|g_y\|_{L^p(I)}$, and define

$$\|f\|_{L^q_y(J,L^p_x(I))} = \|h\|_{L^q(J)}.$$

Thus, for example, when $p \in [1,\infty)$ and $q \in [1,\infty)$ we have

$$\|f\|_{L^q_y(J,L^p_x(I))} = \left(\int_J \left(\int_I |f(x,y)|^p\, dx\right)^{\frac{q}{p}} dy\right)^{\frac{1}{q}}.$$

We define $L^q_y(J, L^p_x(I))$ to be the space of all f for which $\|f\|_{L^q_y(J,L^p_x(I))}$ is finite. In case either I or J is all of $\mathbf{R}$, we omit the reference to I or J. Thus, for example, we refer to $L^q_y(J, L^p_x(\mathbf{R}))$ as $L^q_y(J, L^p_x)$, and to $L^q_y(\mathbf{R}, L^p_x(\mathbf{R}))$ as just $L^q_y(L^p_x)$.

The Fourier transform $\mathcal{F}f(\xi)$ of an integrable function $f(x)$ on $\mathbf{R}$ is defined for $\xi \in \mathbf{R}$ by

$$\mathcal{F}f(\xi) = \int_{-\infty}^{\infty} e^{-i\xi x} f(x)\, dx,$$

and its inverse is given by

$$\mathcal{F}^{-1}F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\xi x}F(\xi)\, d\xi.$$

In the usual way, one can define the Fourier transform and inverse Fourier transform of any tempered distribution in such a way that it agrees with the above definition on L^1.

For any tempered distribution f on $\mathbf{R}$ whose Fourier transform $\mathcal{F}$ is a function, and any $r \in \mathbf{R}$, we define

$$\|f\|_{H^r} = \left(\int_{-\infty}^{\infty} (1 + |\xi|^2)^{r/2}|\mathcal{F}f(\xi)|^2\, d\xi\right)^{1/2}$$

and H^r denotes the Sobolev space of all such tempered distributions f for which $\|f\|_{H^r}$ is finite. For $r \ge 0$, the elements of H^r are functions, and in fact $H^r \subseteq H^0 = L^2$.

We define $\mathcal{S}(\mathbf{R})$ to be the Schwartz space of all complex-valued C^∞ functions on $\mathbf{R}$ such that for every nonnegative integer m and every multi-index α,

$$\sup_{x\in\mathbf{R}} (1 + |x|^2)^{\frac{m}{2}}|D^\alpha u(x)| < \infty$$

If E is a subset of $\mathbf{R}$ then $C_0^\infty(E)$ denotes the space of infinitely differentiable functions with compact support in E.

If X is any Banach space with norm $\|\cdot\|_X$, and $[a,b] \subseteq \mathbf{R}$, we define $C([a,b],X)$ to be the Banach space of all continuous maps $u : [a,b] \to X$ with norm

$$\|u\|_{C([a,b],X)} = \sup_{t\in[a,b]} \|u(t)\|_X.$$

We define $C^1([a,b],X)$ to be the space of all maps $u : [a,b] \to X$ which are differentiable and have continuous derivatives on $[a,b]$.

In general, if $u(x,t)$ is a function of two real variables x and t, we will view u as a map with domain is some interval of values of t, taking values $u(t)$ which are functions of the variable x.

In what follows we will often use C, C_1, and C_2 to stand for various constants, whose value may differ from one line to the next.

2 Statement of main results

We explain the sense in which we interpret the concept of a solution of equation (1.1) or (1.2) in a space, such as L^2, of functions which are not necessarily smooth or even differentiable. For brevity we will refer only to (1.1) in the remarks which follow, but it will be clear that the discussion applies as well, with obvious changes, to (1.2).

Let $D(s)$ be a real-valued, measurable function of $s \in [0,1]$. Then $D(s)$ is well-defined as a real number, at least for almost every s in $[0,1]$. For each such s, we define $T(s)$ as on operator on L^2 by setting

$$\mathcal{F}\left(T(s)[f]\right)(\xi) = \exp\left(-i\xi^2 D(s)\right) \cdot \mathcal{F}f(\xi), \tag{2.1}$$

for all $f \in L^2$. More generally, (2.1) can be used to define $T(s)f$ for all $f \in H^r$, for every real number r. Moreover, since the multiplier $\exp(-i\xi^2 D(s))$ has absolute value 1 for all $\xi \in \mathbf{R}$, it follows easily from Plancherel's theorem that $T(s)$ is unitary on H^r: that is,

$$\|T(s)f\|_{H^r} = \|f\|_{H^r} \tag{2.2}$$

for all $f \in H^r$.

We observe that if u is a solution of (1.1) which is sufficiently well-behaved, say $u \in C(\mathbf{R}, H^1)$, then by Duhamel's principle, u will satisfy the equation

$$u(t) = U(\alpha t)u_0 + i\int_0^t U(\alpha(t-t'))Q(u(t'))\,dt', \tag{2.3}$$

where $Q(u)$ is as defined in (1.4), and $U(t)$ is the group of solution operators for the linear Schrödinger equation $iv_t + v_{xx} = 0$. That is, for each $t \in \mathbf{R}$, we define the unitary operator $U(t) : L^2 \to L^2$ by setting

$$\mathcal{F}(U(t)f)(\xi) = \exp(-i\xi^2 t)\mathcal{F}f(\xi). \tag{2.4}$$

Next we would like to use (2.3) to define a class of solutions of (1.1) which are not necessarily regular enough to lie in H^1.

Definition 2.1. *Suppose $r \geq 0$, $u_0 \in H^r$, and T_1 and T_2 are real numbers with $T_1 < T_2$. We say that $u(x,t) \in C([T_1,T_2],H^r)$ is a strong solution of (1.1) with initial data u_0 if*

1. *for every $t \in [T_1,T_2]$, the map which takes $s \in [0,1]$ to*
$$B(s,t) := g(s)\ T^{-1}(s)\left[|T(s)[u(t)]|^2\ T(s)[u(t)]\right] \tag{2.5}$$
defines a function in $L^1([0,1],H^r)$;

2. *the map which takes $t \in [T_1,T_2]$ to*
$$Q(u(t)) := \int_0^1 B(s,t)\ ds \tag{2.6}$$
defines a function in $C([T_1,T_2],H^r)$; and

3. *the identity*
$$u(t) = U(\alpha(t-T_1))u_0 + i\int_{T_1}^t U(\alpha(t-t'))[Q(u(t'))]\ dt'$$
holds in the sense of equality between elements of $C([T_1,T_2],H^r)$.

It is easy to see (cf. section 1.6 of [C], for example) that strong solutions can be characterized as solutions of (1.1), viewed as a differential equation, with a certain amount of regularity. That is, we have the following proposition.

Proposition 2.2. *Suppose $r \geq 0$ and $u_0 \in H^r$. Then $u(x,t)$ is a strong solution of (1.1) with initial data u_0 if and only if $u \in C([T_1,T_2],H^r) \cap C^1([T_1,T_2],H^{r-2})$, $u(T_1) = u_0$, and $u(t)$ satisfies*
$$i\frac{du}{dt} + \alpha(u(t))_{xx} + Q(u(t)) = 0, \tag{2.7}$$
viewed as a differential equation in H^{r-2}.

To obtain a well-posedness result in H^r for all $r \geq 0$, we will need to make an additional regularity assumption on the function $D(s)$ appearing in the definition of the operator $T(s)$.

Assumption A. *The function $D(s)$ is absolutely continuous on $[0,1]$; and the integrable function $D'(s)$ is piecewise of one sign and is bounded away from zero on $[0,1]$. In other words, there exist $\delta > 0$ and numbers $s_0, s_1, \ldots, s_n$, with $0 = s_0 < s_1 < s_2 < \cdots < s_{n-1} < s_n = 1$, such that for all $j \in \{1,\cdots,n\}$, either $D'(s) \geq \delta$ for almost every $s \in [s_{j-1},s_j]$, or $D'(s) \leq -\delta$ for almost every $s \in [s_{j-1},s_j]$.*

An important role in the theory of the initial-value problem for (1.1) is played by the functional
$$P(u) = \int_{-\infty}^{\infty} |u|^2\ dx$$
and the energy functional
$$E(u) = \int_{-\infty}^{\infty}\left(\alpha\,|u_x|^2 - \frac{1}{2}\int_0^1 g(s)\ |T(s)u|^4\ ds\right)dx.$$

Theorem 2.3. *Suppose $\alpha \in \mathbf{R}$, $g(s)$ is bounded and measurable on $[0,1]$, and $D(s)$ satisfies Assumption A.*

If $r \geq 0$ and $u_0 \in H^r$, then for every $M > 0$, equation (1.1) has a unique strong solution $u \in C([0,M], H^r)$ with initial data u_0. The map $u_0 \mapsto u$ is a locally Lipschitz map from H^r to $C([0,M], H^r)$. The quantity $P(u(t))$ is independent of t for $t \geq 0$; and in case $\alpha = 0$, we can say further that $E(u(t))$ is independent of t for $t \geq 0$.

Also, for every pair of exponents (q,p) which is admissible in the sense of Definition 3.1 below, and every choice of initial data $u_0 \in L^2$, the corresponding solution u belongs to $L^q_t([0,M], L^p_x)$ for every $M > 0$.

Remarks.

(1) Because a strong solution $u(x,t)$ remains a strong solution when transformed to $u(x, t+T)$, we lose no generality by considering solutions with initial data at $t = 0$. Also, since $u(x,-t)$ is a strong solution whenever $u(x,t)$ is, the above result also implies well-posedness backwards in time.

(2) For $r > 1/2$, global well-posedness properties in H^r may be proved by a standard contraction-mapping argument, using the fact that H^r is an algebra. In this case, it is enough to assume only that $D(s)$ is measurable and $g(s)$ is measurable and bounded on $[0,1]$ (see [K] for details).

(3) In case $\alpha = 0$, a well-posedness result is also available in $L^\infty \cap L^2$ (cf. Theorem 2.11 of [K]).

(4) In the proof of Theorem 2.3, the fact that $P(u(t))$ is conserved is crucial, whereas the fact that $E(u(t))$ is conserved is not needed. However, the fact that $E(u(t))$ is conserved when $\alpha = 0$ is important for the theory of stability of solitary waves, so we include it here.

For the 1DCR equation (1.2), one has a similar definition of strong solution, in which the obvious changes are made in definitions (2.5) and (2.6), and a similar well-posedness result. Define the energy functional E_2 for (1.2) by

$$E_2(u) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |U(s)u|^6 \, ds \, dx. \tag{2.8}$$

Theorem 2.4. *If $r \geq 0$ and $u_0 \in H^r$, then for every $M > 0$, equation (1.2) has a unique strong solution $u \in C([0,M], H^r)$ with initial data u_0. The map $u_0 \mapsto u$ is a locally Lipschitz map from H^r to $C([0,M], H^r)$. The quantities $P(u(t))$ and $E_2(u(t))$ are independent of t for $t \geq 0$.*

Also, for every pair of exponents (q,p) which is admissible in the sense of Definition 3.1 below, and every choice of initial data $u_0 \in L^2$, the corresponding solution u belongs to $L^q_t([0,M], L^p_x)$ for every $M > 0$.

We will now state a result showing that the condition $r \geq 0$ in Theorem 2.4 is sharp. Let us denote the nonlinear term in (1.2) by

$$Q_2(u) = \int_{-\infty}^{\infty} U(s)^{-1} \left[|U(s)[u]|^4 U(s)[u] \right] \, ds. \tag{2.9}$$

For every $\omega > 0$, every $\beta > 0$, and every $N \in \mathbf{R}$, equation (1.2) has solutions of the form

$$u(x,t) = \beta\omega e^{i\beta^4\omega^2 t} e^{iNx} \varphi(\omega x), \tag{2.10}$$

where $\varphi(x)$ is a solution of

$$Q_2(\varphi) = \varphi. \tag{2.11}$$

Concerning the existence of solutions of (2.11), we have the following.

Proposition 2.5. *For every $b > 0$ and every $x_0 \in \mathbf{R}$ and $\theta \in [0, 2\pi]$, the function*

$$\varphi(x) = \left(\frac{4b\sqrt{3}}{\pi}\right)^{1/4} e^{-b(x-x_0)^2 + i\theta x} \tag{2.12}$$

is a solution of (2.11).

Proof. According to Theorem 1.5 of [HZ], every function of the form $g(x) = Ae^{-b(x-x_0)^2}$ with $A \in \mathbf{C}$ and $b > 0$ is a maximizer for the functional

$$S(f) = \frac{\|U(t)f\|_{L_t^6(L_x^6)}}{\|f\|_{L^2}}$$

over all functions $f \in L^2$. Therefore g is a stationary point of S; that is, $\nabla S(g) = 0$, which implies that

$$Q_2(g) = \lambda g \tag{2.13}$$

for some $\lambda \in \mathbf{R}$. Taking the L^2 inner product of (2.13) with g, we obtain that

$$\langle Q_2(g), g\rangle = \lambda \|g\|_{L^2}^2 = \lambda |A|^2 \sqrt{\frac{\pi}{2b}}. \tag{2.14}$$

On the other hand, from Theorem 1.3(a) in [HZ] we have that

$$\begin{aligned} \langle Q_2(g), g\rangle &= E_2(g) \\ &= |A|^6 \frac{1}{2\sqrt{3}} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-2b(x_1^2+x_2^2+x_3^2)}\, dx_1\, dx_2\, dx_3 \\ &= |A|^6 \frac{1}{2\sqrt{3}} \left(\frac{\pi}{2b}\right)^{3/2}. \end{aligned} \tag{2.15}$$

Combining (2.14) and (2.15), we find that if $A = \left(4b\sqrt{3}/\pi\right)^{1/4} e^{i\theta x}$, then $\lambda = 1$. Since $g = \phi$ in this case, (2.13) implies (2.11). $\square$

Remark. All the solutions of (2.11) given in (2.12) have the same L^2 norm: namely, $\|\varphi\|_{L^2} = 12^{1/8}$. It is known (cf. [HZ]) that the functions given in (2.12) represent all the solutions of (2.11) which are minimizers of $-S(f)$, the so-called ground state solutions of (2.11). An interesting question is whether other solutions of (2.11) exist besides these ground states.

Theorem 2.6. *Suppose $r < 0$, and $M > 0$. Let φ be any of the solutions of* (2.11) *given in* (2.12). *Then for every $\delta > 0$, there exist two solutions $u(x,t)$ and $v(x,t)$ of* (1.2) *in $C([0,M], H^r)$ for which*

$$\|u(x,0)\|_{H^r} \le 2\|\varphi\|_{L^2} \text{ and } \|v(x,0)\|_{H^r} \le 2\|\varphi\|_{L^2}, \tag{2.16}$$

$$\|u(x,0)-v(x,0)\|_{H^r} < \delta, \tag{2.17}$$

and

$$\|u(x,M)-v(x,M)\|_{H^r} \geq 2^{r-2}\|\varphi\|_{L^2}. \tag{2.18}$$

Remark. In fact, the only properties of φ that we need to prove Theorem 2.6, besides that it is a solution of (2.11), are that $|\xi|^2\mathcal{F}\varphi(\xi)$ is bounded for $\xi \in \mathbf{R}$ and $\int_{-\infty}^{\infty} |\xi|\,|(\mathcal{F}\varphi)'(\xi)|^2 \; d\xi$ is finite. Note that these conditions are satisfied by the solutions φ given in (2.12), for which we have that $\mathcal{F}\varphi(\xi) = (4\pi\sqrt{3}/b)^{1/4}e^{-i(\xi-\theta)^2/4b - i(\xi-\theta)x_0}$.

Remark. If we denote by $B(R)$ the ball of radius R centered at the origin in H^r, it follows from Proposition 2.5 and Theorem 2.6 that no uniformly continuous map taking initial data to solutions of (1.2) can be defined from $B(2(12^{1/8}))$ into $C([0,M],H^r)$. We do not know, however, whether there exists some smaller ball $B(R)$ of positive radius on which such a uniformly continuous solution map can be defined.

3 Strichartz and Sobolev estimates

First we recall some well-known Strichartz estimates for the Schrödinger operators $U(t)$.

Definition 3.1. *We say that (q,p) is an admissible pair of exponents if $2 \leq p \leq \infty$, $4 \leq q \leq \infty$, and*

$$\frac{2}{q} = \frac{1}{2} - \frac{1}{p}\ .$$

Lemma 3.2. *Suppose (q,p) is an admissible pair of exponents. Then there exists $C > 0$ such that for all $u \in L^2$, $U(t)u$ belongs to $L^q_t(L^p_x)$ and*

$$\|U(t)u\|_{L^q_t(L^p_x)} \leq C\|u\|_{L^2}.$$

Moreover, $U(t)u \in C(\mathbf{R}, L^2)$. More generally, if $u \in H^r$ for some $r \in \mathbf{R}$, then $U(t)u \in C(\mathbf{R}, H^r)$, and for all $t \in \mathbf{R}$,

$$\|U(t)u\|_{H^r} = \|u\|_{H^r}. \tag{3.1}$$

Lemma 3.3. *Suppose (q,p) and (γ,ρ) are two admissible pairs of exponents, and let γ' and ρ' be the dual exponents to γ and ρ: i.e., $\gamma' = \gamma/(\gamma-1)$ and $\rho' = \rho/(\rho-1)$. Suppose $M > 0$. Then for all $f(x,t) \in L^{\gamma'}_t([0,M], L^{\rho'}_x)$, the function*

$$v(t) = \int_0^t U(t-t')[f(t')]\; dt'$$

belongs to $L^q_t([0,M], L^p_x)$, and

$$\|v\|_{L^q_t([0,M],L^p_x)} \leq C\|f\|_{L^{\gamma'}_t([0,M],L^{\rho'}_x)},$$

where $C > 0$ is a constant which is independent of M.

Moreover, if $f \in C([0,M], H^r)$ for some $r \in \mathbf{R}$, then $v \in C([0,M], H^r)$.

For proofs of Lemmas 3.2 and 3.3 the reader may consult, for example, Section 2.3 of [C].

Next, we observe that the Strichartz estimates in Lemma 3.2 also hold for the semigroup $T(s)$, provided Assumption A holds. This fact can be viewed as an instance of the general Strichartz estimates proved in [KT]; for a similar result see also Lemma 2.5 of [ASS]. For completeness we give the details of the proof here.

Lemma 3.4. *Suppose $D(s)$ satisfies Assumption A, and suppose (q,p) is an admissible pair of exponents. Then there exists $C > 0$ such that for all $u \in L^2$, we have that $T(s)u(x)$ belongs to $L^q_s([0,1], L^p_x)$, and*

$$\|T(s)u\|_{L^q_s([0,1],L^p_x)} \leq C\|u\|_{L^2}. \tag{3.2}$$

Moreover, $T(s)u \in C([0,1], L^2)$. More generally, if $u \in H^r$ for some $r \in \mathbf{R}$, then $T(s)u \in C([0,1], H^r)$, and for all $s \in [0,1]$,

$$\|T(s)u\|_{H^r} = \|u\|_{H^r}.$$

Proof. Note first that since $\mathcal{S}(\mathbf{R})$ is dense in $L^2(\mathbf{R})$, it is enough to prove the result for $u \in \mathcal{S}(\mathbf{R})$. In this case, $T(s)u(x)$ is a continuous function of $(s,x) \in [0,1] \times \mathbf{R}$, and therefore measurable on $[0,1] \times \mathbf{R}$. Hence we need only prove the estimate (3.2).

Recall (cf. Lemma 2.2.4 of [C], or sections 1.3 and 4.1 of [LP]) that the linear Schrödinger solution operator $U(t)$ can be represented as a convolution operator: for every $t > 0$ and every $\phi \in \mathcal{S}(\mathbf{R})$, one has

$$U(t)\phi(x) = \frac{1}{2\pi}\frac{1}{\sqrt{4\pi i t}}\int_{-\infty}^{\infty} \exp\left[-i\left(\frac{(x-y)^2}{4t}\right)\right]\phi(y)\, dy, \tag{3.3}$$

where we choose the branch of the complex square root so that $\sqrt{4\pi i t}$ has positive imaginary part.

Next, observe that by writing

$$\begin{aligned}\|T(s)u\|_{L^q_s((0,1),L^p_x)} &= \left(\sum_{j=1}^{n}\int_{s_{j-1}}^{s_j} \|T(s)u\|^q_{L^p}\, ds\right)^{1/q} \\ &\leq C\sum_{j=1}^{n} \|T(s)u\|_{L^q_s((s_{j-1},s_j),L^p_x)},\end{aligned}$$

and taking Assumption A into account, one sees that it is enough to prove that the estimate (3.2) holds when $[0,1]$ is replaced by an arbitrary finite interval $[a,b]$, under the assumption that $D(s)$ is absolutely continuous on $[a,b]$ and

$$\text{either } D'(s) \geq \delta \text{ a.e. on } [a,b], \text{ or } D'(s) \leq -\delta \text{ a.e. on } [a,b]. \tag{3.4}$$

Let q' and p' be the dual exponents of q and p, with $\frac{1}{q'} + \frac{1}{q} = 1$ and $\frac{1}{p'} + \frac{1}{p} = 1$, and let $B = \{\phi \in L^{q'}_s((a,b), L^{p'}_x) : \|\phi\|_{L^{q'}_s((a,b),L^{p'}_x)} \leq 1\}$. From duality we have that

$$\|T(s)u\|_{L^q_s((a,b),L^p_x)} = \sup\left\{\int_a^b \langle T(s)u, \phi(s)\rangle\ ds : \phi \in B\right\}. \tag{3.5}$$

In fact, by a density argument we can take the supremum in (3.5) to be over all $\phi \in B$ such that $\phi(s) \in \mathcal{S}(\mathbf{R})$ for all $s \in [0,1]$.

Now

$$\begin{aligned}
\int_a^b \langle T(s)u, \phi(s)\rangle \; ds &= \int_a^b \langle u, T^*(s)\phi(s)\rangle \; ds \\
&= \left\langle u, \int_a^b T^*(s)\phi(s)\; ds \right\rangle \\
&\leq \|u\|_{L^2} \left\| \int_a^b T^*(s)\phi(s)\; ds \right\|_{L^2},
\end{aligned}$$

where $T^*(s) = T^{-1}(s)$ is the adjoint of the unitary operator $T(s)$ on L^2. So from (3.5) it follows that to prove the theorem, it is enough to show that there exists $C > 0$ so that

$$\left\| \int_a^b T^*(s)\phi(s)\; ds \right\|_{L^2} \leq C \tag{3.6}$$

for all $\phi \in B$ such that $\phi(s) \in \mathcal{S}(\mathbf{R})$ for all $s \in [0,1]$.

We have

$$\begin{aligned}
\left\| \int_a^b T^*(s)\phi(s)\; ds \right\|_{L^2}^2 &= \left\langle \int_a^b T^*(s)\phi(s)\; ds, \int_a^b T^*(t)\phi(t)\; dt \right\rangle \\
&= \int_a^b \langle \phi(t), \theta_\phi(s)\rangle \; ds,
\end{aligned}$$

where

$$\theta_\phi(s) = \int_a^b T(s)T^*(t)\phi(t)\; dt.$$

From two applications of Hölder's Inequality it follows that

$$\begin{aligned}
\int_a^b \langle \phi(t), \theta_\phi(s)\rangle \; ds &\leq \int_a^b \|\phi(s)\|_{L^{p'}} \|\theta_\phi(s)\|_{L^p}\; ds \\
&\leq \|\phi\|_{L_s^{q'}((a,b),L_x^{p'})} \|\theta_\phi\|_{L_s^q((a,b),L_x^p)} \leq \|\theta_\phi\|_{L_s^q((a,b),L_x^p)},
\end{aligned}$$

so to prove (3.6) it is enough to obtain a uniform bound on $\|\theta_\phi\|_{L_s^q((a,b),L_x^p)}$.

For each $s \in (0,1)$ we have

$$\|\theta_\phi(s)\|_{L^p} = \left\| \int_a^b T(s)T^*(t)\phi(t)\; dt \right\|_{L^p} \leq \int_a^b \|T(s)T^*(t)\phi(t)\|_{L^p}\; dt.$$

From the definition of $T(s)$ we see that

$$\mathcal{F}\left(T(s)T^*(t)\phi(t)\right) = \exp\left(-i\omega^2(D(s)-D(t))\right) \mathcal{F}\phi(t),$$

so

$$T(s)T^*(t)\phi(t) = U(D(s)-D(t))\phi(t),$$

and hence from (3.3) we obtain, for each $s, t \in [0,1]$,

$$
\begin{aligned}
&\left(T(s)T^*(t)\phi(t)\right)(x) = \\
&= \frac{1}{2\pi} \frac{1}{\sqrt{4\pi i(D(s)-D(t))}} \int_{-\infty}^{\infty} \exp\left(\frac{-i|x-y|^2}{4(D(s)-D(t))}\right) \phi(t,y)dy.
\end{aligned}
$$

Taking the supremum over $x \in \mathbf{R}$ gives

$$
\|T(s)T^*(t)\phi(t)\|_{L^\infty} \leq \frac{C\|\phi(t)\|_{L^1}}{|D(s)-D(t)|^{1/2}}. \tag{3.7}
$$

On the other hand, since T is unitary, we have

$$
\|T(s)T^*(t)\phi(t)\|_{L^2} = \|\phi(t)\|_{L^2}. \tag{3.8}
$$

From (3.7), (3.8) and the Riesz-Thorin Interpolation Theorem (see, for example, Theorem 2.1 of [LP]), we get

$$
\|T(s)T^*(t)\phi(t)\|_{L^p} \leq \left(\frac{C}{\sqrt{|D(s)-D(t)|}}\right)^{1-(2/p)} C^{2/p}\|\phi(t)\|_{L^{p'}},
$$

for some constant C independent of ϕ. So

$$
\|\theta_\phi(s)\|_{L^p} \leq C \int_a^b \frac{\|\phi(t)\|_{L^{p'}}}{|D(s)-D(t)|^{(1/2)-(1/p)}}\, dt.
$$

But from our assumption (3.4) on $D(s)$, it follows that

$$
|D(s)-D(t)| \geq \delta|s-t|.
$$

Therefore

$$
\|\theta_\phi(s)\|_{L^p} \leq C\delta^{(1/p)-(1/2)}\, w(s), \tag{3.9}
$$

where

$$
w(s) = \int_a^b \frac{\|\phi(t)\|_{L^{p'}}}{|s-t|^{(1/2)-(1/p)}}\, dt.
$$

Now, in case $p > 2$, we define

$$
\beta = \frac{1}{2} + \frac{1}{p} \in (0,1),
$$

and observe that since (q,p) is an admissible pair, we have that

$$
\frac{1}{q} = \frac{1}{q'} - \beta.
$$

Therefore, the Hardy-Littlewood-Sobolev inequality (see, e.g., Theorem 2.6 of [LP]) for fractional integrals of order β, when applied to the function $w(s)$, yields the estimate

$$
\|w\|_{L^q(a,b)} \leq C\left(\int_a^b \|\phi(s)\|_{L^{p'}}^{q'}\, ds\right)^{1/q'} = C\|\phi\|_{L_s^{q'}((a,b),L_x^{p'})} = C.
$$

Together with (3.9), this gives the desired bound on $\|\theta_\phi\|_{L^q_s((a,b),L^p_x)}$, and therefore proves (3.2) in the case when $p > 2$.

To prove (3.2) in the remaining case, when $p = 2$ and $q = \infty$, observe first that if, say, $u \in \mathcal{S}(\mathbf{R})$, then since

$$T(s)u(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega x} e^{-i\omega^2 D(s)} \mathcal{F}u(\omega)\, d\omega$$

and $D(s)$ is continuous on $[0, 1]$, it follows from the dominated convergence theorem that $T(s)u(x)$ is a continuous function of (x, s) on $\mathbf{R} \times [0, 1]$, and lies in $C([0, 1], L^2)$. Also the estimate (3.2) for $p = 2$ and $q = \infty$ is a special case of (2.2). Then, for general $u \in L^2$, by approximating u by functions in $\mathcal{S}(\mathbf{R})$ and passing to the limit we can conclude that $T(s)u \in C([0, 1], L^2)$ and (3.2) still holds.

To prove the final assertions of the Lemma concerning H^r, one simply observes that a tempered distribution u is in H^r if and only if $(1+\partial_x^2)^{r/2}u \in L^2$, where $\mathcal{F}((1+\partial_x^2)^{r/2}u)(\omega) = (1 + \omega^2)^{r/2}\mathcal{F}u(\omega)$. The desired results then follow from what has already been proved and the fact that $T(s)$ commutes with $(1 + \partial_x^2)^{r/2}$. □

For the next estimate we will need to use the following product estimate for Sobolev norms. For a proof the reader may consult, for example, Lemma A.8 of [T].

Lemma 3.5. *Suppose $r \geq 0$. Then there exists $C > 0$ such that for all $u, v \in H^r \cap L^\infty$,*

$$\|uv\|_{H^r} \leq C \left(\|u\|_{L^\infty}\|v\|_{H^r} + \|v\|_{L^\infty}\|u\|_{H^r}\right).$$

The following multilinear estimate for Q was already proved in [Ku] in the case where $D(s) = s$ and $r = 0$. See also [HL], where it is proved for $r = 0$ under even more general assumptions on $D(s)$ than the ones used here.

Lemma 3.6. *Suppose $r \geq 0$. Suppose $u_1, u_2, u_3 \in H^r$, and for $s \in [0, 1]$ define*

$$B(s) := g(s)\ T^{-1}(s) \left[T(s)u_1 \cdot \overline{T(s)u_2} \cdot T(s)u_3\right].$$

Then $B(s) \in L^1([0, 1], H^r)$, and if we define

$$Q(u_1, u_2, u_3) := \int_0^1 B(s)ds, \tag{3.10}$$

then we have

$$\begin{aligned}\|Q(u_1, u_2, u_3)\|_{H^r} \leq C(&\|u_1\|_{H^r}\|u_2\|_{L^2}\|u_3\|_{L^2} + \|u_1\|_{L^2}\|u_2\|_{H^r}\|u_3\|_{L^2}\\ &+ \|u_1\|_{L^2}\|u_2\|_{L^2}\|u_3\|_{H^r}),\end{aligned} \tag{3.11}$$

with C independent of u_1, u_2, and u_3.

Proof. We may assume that $u_i \in \mathcal{S}(\mathbf{R})$ for each i, so that all functions involved are measurable (even continuous). For each i, we have by Lemma 3.4 that $T(s)u_i \in L^4_s([0, 1], L^\infty_x)$, with

$$\|u_i\|_{L^4_s([0,1],L^\infty_x)} \leq C\|u_i\|_{L^2}.$$

In particular $T(s)u_i \in L^\infty$ for almost every $s \in [0,1]$. Also, by Lemma 3.4, $T(s)u_i \in H^r$ for all $s \in [0,1]$. Therefore we may apply Lemma 3.5 to the product $T(s)u_1 \cdot \overline{T(s)u_2} \cdot T(s)u_3$ for almost every $s \in [0,1]$. There results the estimate

$$\begin{aligned}\|T(s)u_1 \cdot \overline{T(s)u_2} \cdot T(s)u_3\|_{H^r} \le C(&\|T(s)u_1\|_{H^r}\|T(s)u_2\|_{L^\infty}\|T(s)u_3\|_{L^\infty} \\ &+ \|T(s)u_1\|_{L^\infty}\|T(s)u_2\|_{H^r}\|T(s)u_3\|_{L^\infty} \\ &+ \|T(s)u_1\|_{L^\infty}\|T(s)u_2\|_{L^\infty}\|T(s)u_3\|_{H^r}).\end{aligned}$$

Since $g(s)$ is bounded and $T^{-1}(s)$ is a unitary operator on H^r, it follows that

$$\begin{aligned}\|Q(u_1,u_2,u_3)\|_{H^r} \le C \Bigg(&\int_0^1 \|T(s)u_1\|_{H^r}\|T(s)u_2\|_{L^\infty}\|T(s)u_3\|_{L^\infty}\, ds \\ &+ \int_0^1 \|T(s)u_1\|_{L^\infty}\|T(s)u_2\|_{H^r}\|T(s)u_3\|_{L^\infty}\, ds \\ &+ \int_0^1 \|T(s)u_1\|_{L^\infty}\|T(s)u_2\|_{L^\infty}\|T(s)u_3\|_{H^r}\, ds \Bigg).\end{aligned} \tag{3.12}$$

Using Lemma 3.4 and Holder's inequality, we have, for the first integral on the right-hand side of (3.12), the estimate

$$\begin{aligned}&\int_0^1 \|T(s)u_1\|_{H^r}\|T(s)u_2\|_{L^\infty}\|T(s)u_3\|_{L^\infty}\, ds \\ &\quad \le C\|u_1\|_{H^r}\|T(s)u_2\|_{L^4_s([0,1],L^\infty_x)}\|T(s)u_3\|_{L^4_s([0,1],L^\infty_x)} \\ &\quad \le C\|u_1\|_{H^r}\|u_2\|_{L^2}\|u_3\|_{L^2};\end{aligned}$$

and similar estimates hold for the other two integrals in (3.12). □

Lemma 3.7. *Suppose $r \ge 0$. Let $Q(u_1,u_2,u_3)$ be as defined in (3.10), and for $u \in H^r$, define*

$$Q(u) := Q(u,u,u) = \int_0^1 g(s)\ T^{-1}(s)\left[|T(s)u|^2\ T(s)u\right]\ ds.$$

For all $u, v \in H^r$ we have

$$\|Q(u)\|_{H^r} \le C\|u\|_{H^r}\|u\|^2_{L^2} \tag{3.13}$$

and

$$\|Q(u)-Q(v)\|_{H^r} \le C\|u-v\|_{H^r}\max(\|u\|_{L^2},\|v\|_{L^2})\max(\|u\|_{H^r},\|v\|_{H^r}) \tag{3.14}$$

where C is independent of u and v.

Proof. The estimate (3.13) follows immediately from (3.11); and so does (3.14), once we observe that

$$Q(u)-Q(v) = Q(u-v,u,u)+Q(v,u-v,u)+Q(v,v,u-v).$$

□

We will need below the following fact concerning $E(u)$.

Lemma 3.8. *If $u \in L^2$, then*

$$\langle Q(u), u\rangle = \int_{-\infty}^{\infty}\int_0^1 g(s)|T(s)u|^4\ ds\ dx. \tag{3.15}$$

The functional $E(u)$ is continuous on H^1; and, when $\alpha = 0$, is continuous on L^2 as well.

Proof. Equation (3.15) is obtained by writing

$$\begin{aligned}
\langle Q(u), u\rangle &= \int_{-\infty}^{\infty}\left(\int_0^1 g(s)T^{-1}(s)\left[|T(s)u|^2T(s)u\right]\ ds\right)\overline{u}\ dx \\
&= \int_0^1 g(s)\left\langle T^{-1}(s)\left[|T(s)u|^2T(s)u\right], u\right\rangle\ ds \\
&= \int_0^1 g(s)\left\langle |T(s)u|^2T(s)u, T(s)u\right\rangle\ ds \\
&= \int_{-\infty}^{\infty}\int_0^1 g(s)|T(s)u|^4\ ds\ dx.
\end{aligned}$$

Here we have used using Fubini's theorem, which is justified by Lemma 3.7, and the fact that $T(s)$ is unitary on L^2.

It follows that

$$E(u) = \alpha\|u_x\|_{L^2}^2 - \frac{1}{2}\langle Q(u), u\rangle,$$

for all $u \in H^1$; and, when $\alpha = 0$, we have

$$E(u) = -\frac{1}{2}\langle Q(u), u\rangle$$

for all $u \in L^2$. The assertions about the continuity of $E(u)$ follow from the fact that for all $u, v \in L^2$ we have, using (3.13) and (3.14),

$$\begin{aligned}
|\langle Q(u), u\rangle - \langle Q(v), v\rangle| &\le C\left(|\langle Q(u) - Q(v), u\rangle| + |\langle Q(v), u - v\rangle|\right) \\
&\le C\|Q(u) - Q(v)\|_{L^2}\|u\|_{L^2} + C\|Q(v)\|_{L^2}\|u - v\|_{L^2} \\
&\le C\|u - v\|_{L^2}\max\left(\|u\|_{L^2}^3 + \|v\|_{L^2}^3\right).
\end{aligned}$$

□

Finally we observe that estimates similar to the ones obtained above hold for the nonlinear term $Q_2(u)$ in the 1DCR equation, defined above in (2.9).

Lemma 3.9. *Suppose $r \ge 0$. Then for all $u, v \in H^r$ we have*

$$\|Q_2(u)\|_{H^r} \le C\|u\|_{H^r}\|u\|_{L^2}^4, \tag{3.16}$$

and

$$\|Q_2(u) - Q_2(v)\|_{H^r} \le C\|u - v\|_{H^r}\max(\|u\|_{L^2}^3, \|v\|_{L^2}^3)\max(\|u\|_{H^r}, \|v\|_{H^r}); \tag{3.17}$$

where C is independent of u and v.

Moreover, for all $u \in L^2$, the functional $E_2(u)$ defined in (2.8) *is given by*

$$E_2(u) = \langle Q_2(u), u \rangle,$$

and hence E_2 represents a continuous map from L^2 to $\mathbf{R}$.

Proof. First observe that taking $q = \infty$ and $p = 4$ in Lemma (3.2) gives the estimate

$$\|U(s)u\|_{L^4(L^\infty)} \leq C\|u\|_{L^2} \tag{3.18}$$

for all $u \in L^2$. If $u_i \in H^r$ for $i = 1, 2, 3, 4, 5$, we define

$$Q_2(u_1, u_2, u_3, u_4, u_5) :=$$
$$:= \int_{-\infty}^{\infty} U^{-1}(s) \left[U(s)u_1 \cdot \overline{U(s)u_2} \cdot U(s)u_3 \cdot \overline{U(s)u_4} \cdot U(s)u_5 \right] \, ds,$$

then arguing as in Lemma (3.6), using (3.18) and Holder's inequality, we obtain the estimate

$$\|Q_2(u_1, u_2, u_3, u_4, u_5)\|_{H^r} \leq C \sum \|u_{i_1}\|_{H^r} \|u_{i_2}\|_{L^2} \|u_{i_3}\|_{L^2} \|u_{i_4}\|_{L^2} \|u_{i_5}\|_{L^2},$$

where the sum is over all the permutations $(i_1, i_2, i_3, i_4, i_5)$ of $(1, 2, 3, 4, 5)$. The estimates (3.16) and (3.17) then follow as in the proof of Lemma 3.7, and the assertions concerning E_2 follow as in the proof of Lemma 3.8. □

4 Well-posedness for DMNLS and 1DCR

In this section we prove Theorems 2.3 and 2.4. The proofs proceed by standard arguments once the estimates in the previous section have been obtained, but for the reader's convenience we give the argument here in detail.

We denote by $B_{M,a,r}$ the closed ball of radius a centered at the origin in $C([0, M], H^r)$; that is,

$$B_{M,a,r} = \left\{ u \in C([0, M], H^r) : \|u\|_{C([0,M],H^r)} \leq a \right\}.$$

Lemma 4.1. *Suppose $r \geq 0$ and $K > 0$. For every $a \in [2K, \infty)$, there exists $M > 0$ such that if $u_0 \in H^r$ satisfies $\|u_0\|_{H^r} \leq K$, and $M' \in (0, M]$, then there is a unique strong solution of* (1.1) *in $B_{M',a,r}$ with initial data u_0.*

Proof. Fix $u_0 \in H^r$ such that $\|u_0\|_{H^r} \leq K$. For each $M > 0$ we claim that we can define $\Phi : C([0, M], H^r) \to C([0, M], H^r)$ by setting, for $u \in C([0, M], H^r)$ and $t \in [0, M]$,

$$\Phi(u)(t) = U(\alpha t)u_0 + i \int_0^t U(\alpha(t - t'))[Q(u(t'))] \, dt'. \tag{4.1}$$

Indeed, from (3.14) we know that $Q(u(t)) \in C([0, M], H^r)$, and from Lemmas 3.2 and 3.3 it then follows that $\Phi(u)$ is well-defined as an element of $C([0, M], H^r)$. Moreover, for all $t \in [0, M]$, we have that

$$\begin{aligned} \|\Phi(u)(t)\|_{H^r} &\leq \|U(\alpha t)u_0\|_{H^r} + C \int_0^t \|U(\alpha(t - t'))[Q(u(t'))]\|_{H^r} \, dt' \\ &\leq \|u_0\|_{H^r} + C \int_0^t \|u(t')\|_{H^r}^3 \, dt', \end{aligned} \tag{4.2}$$

where we have used (3.1) and (3.13).

Further, if $u, v \in C([0,M], H^r)$, then using (3.14) we have that, for all $t \in [0,M]$,

$$\begin{aligned}
\|\Phi(u)(t) - \Phi(v)(t)\|_{H^r} &= \left\| \int_0^t U(\alpha(t-t'))[Q(u)(t') - Q(v)(t')]\ dt' \right\|_{H^r} \\
&\le \int_0^t \|Q(u(t')) - Q(v(t'))\|_{H^r}\ dt' \\
&\le C \int_0^t \|u(t') - v(t')\|_{H^r} \max(\|u(t')\|_{H^r}^2, \|v(t')\|_{H^r}^2)\ dt'.
\end{aligned} \tag{4.3}$$

From (4.2) and (4.3) it follows that if $a > 0$, then for all $u, v \in B_{M,a,r}$ we have

$$\|\Phi(u)\|_{C([0,M],H^r)} \le K + CMa^3 \tag{4.4}$$

and

$$\|\Phi(u) - \Phi(v)\|_{C([0,M],H^r)} \le CMa^2 \|u - v\|_{C([0,M],H^r)}. \tag{4.5}$$

Now suppose $a \ge 2K$, choose $M = 1/(2Ca^2)$, and suppose $0 < M' \le M$. For all $u, v \in B_{M',a,r}$, we have from (4.4) and (4.5) that

$$\|\Phi(u)\|_{C([0,M'],H^r)} \le \frac{a}{2} + \frac{a}{2} = a$$

and

$$\|\Phi(u) - \Phi(v)\|_{C([0,M'],H^r)} \le \frac{1}{2} \|u - v\|_{C([0,M'],H^r)}.$$

Therefore Φ defines a contraction from the closed ball $B_{M',a,r}$ into itself, and so it follows from the Banach contraction mapping theorem that Φ has a unique fixed point in $B_{M',a,r}$. This fixed point is a strong solution of (1.1) with initial data u_0, and since every strong solution with initial data u_0 is also a fixed point of Φ, then there exists a unique strong solution in $B_{M',a,r}$ with initial data u_0. ☐

Lemma 4.2. *Suppose $r \ge 0$ and $K > 0$, and suppose $u_0, v_0 \in H^r$ with $\|u_0\|_{H^r} \le K$ and $\|v_0\|_{H^r} \le K$. Let $a = 2K$ and let M be as defined in the statement of Lemma 4.1, and let u and v be the unique strong solutions in $B_{M,2K,r}$ with initial data u_0 and v_0, respectively, given by Lemma 4.1. Then*

$$\|u - v\|_{C([0,M],H^r)} \le 2\|u_0 - v_0\|_{H^r}. \tag{4.6}$$

Proof. Defining Φ on $C([0,M], H^r)$ by (4.1), we see from the proof of Lemma 4.1 that

$$\|\Phi(u) - \Phi(v)\|_{C([0,M],H^r)} \le \frac{1}{2} \|u - v\|_{C([0,M],H^r)}. \tag{4.7}$$

Define the map $\Psi : C([0,M], H^r) \times H^r \to C([0,M], H^r)$ by

$$\Psi(u, w) = U(\alpha t)w + i \int_0^t U(\alpha(t-t'))[Q(u(t'))]\ dt'.$$

Then $u = \Psi(u, u_0)$ and $v = \Psi(v, v_0)$, and so for each $t \in [0, M]$,

$$\begin{aligned}\|u(t)-v(t)\|_{H^r} &= \|\Psi(u,u_0)(t) - \Psi(v,v_0)(t)\|_{H^r}\\ &\le \|\Psi(u,u_0)(t) - \Psi(v,u_0)(t)\|_{H^r} + \|\Psi(v,u_0)(t) - \Psi(v,v_0)(t)\|_{H^r}.\end{aligned}$$

But $\|\Psi(v,u_0)(t) - \Psi(v,v_0)(t)\|_{H^r} = \|U(\alpha t)[u_0 - v_0]\|_{H^r} = \|u_0 - v_0\|_{H^r}$, and since $\Psi(u,u_0)(t) - \Psi(v,u_0)(t) = \Phi(u)(t) - \Phi(v)(t)$, it follows from (4.7) that $\|\Psi(u,u_0)(t) - \Psi(v,u_0)(t)\|_{H^r} \le \frac{1}{2}\|u - v\|_{C([0,M],H^r)}$. Therefore

$$\|u(t) - v(t)\|_{H^r} \le \|u_0 - v_0\|_{H^r} + \frac{1}{2}\|u - v\|_{C([0,M],H^r)},$$

and since this has been proved for all $t \in [0, M]$, (4.6) follows. □

Lemma 4.3. *Suppose $u_0 \in L^2$ and $M > 0$. Then there cannot be two different strong solutions of (1.1) in $C([0,M], L^2)$ with initial data u_0.*

Proof. Suppose u and v are two strong solutions in $C([0,M], L^2)$ with the same initial data u_0, and let

$$T = \sup\{t \in [0,M] : u(t') = v(t') \quad \text{for all } t' \in [0,t]\}.$$

By continuity, we have that $u(T) = v(T)$.

We want to show that $T = M$. If, to the contrary, $T < M$, then it follows from the definition of T that for every $\epsilon \in (0, M - T)$, there exists $t \in [T, T + \epsilon]$ such that $u(t) \ne v(t)$. Define $u_1 = u(T) = v(T)$. From Lemma 4.1 we know that if we define $a_1 = 2\|u_1\|_{L^2}$, then there exists $M_1 > 0$ such that for every $\epsilon \in (0, M_1]$, (1.1) has a unique strong solution in $B_{\epsilon,a_1,0}$ with initial data u_1. Choose $\epsilon > 0$ so small that $\epsilon < \min(M_1, M - T)$, and $\max(\|u(t) - u_1\|_{L^2}, \|v(t) - u_1\|_{L^2}) \le a_1$ for all $t \in [T, T + \epsilon]$. Then $\tilde{u}(t) := u(t - T)$ and $\tilde{v}(t) := v(t - T)$ are two distinct strong solutions in $B_{\epsilon,a_1,0}$ with initial data u_1, giving a contradiction. This shows that we must have $T = M$, and hence $u = v$ in $C([0,M], L^2)$. □

For given $u_0 \in H^r$, with $r \ge 0$, we define $M(u_0, r)$ to be the supremum of the set of all $M > 0$ such that there exists a strong solution of (1.1) in $C([0,M], H^r)$ with initial data u_0. From Lemma 4.1 we have that $M(u_0, r) > 0$; and since every strong solution in $C([0,M], H^r)$ is also a strong solution in $C([0,M], L^2)$, it follows from Lemma 4.3 that two solutions defined on different time intervals $[0, M_1]$ and $[0, M_2]$ must agree on the smaller of the two intervals. Therefore there is a well-defined function $u(x,t)$ defined for $t \in [0, M(u_0, r))$ such that for every $M \in (0, M(u_0, r))$, u is the unique strong solution of (1.1) in $C([0,M], H^r)$ with initial data u_0. Moreover, if $M(u_0, r) < \infty$, then $\lim_{t \nearrow M(u_0,r)} \|u(t)\|_{H^r} = \infty$, for otherwise we obtain a contradiction by choosing as initial data $u(M)$ with M sufficiently close to $M(u_0, r)$, and using Lemma 4.1 to extend the solution $u(t)$ to a time interval $[0, M + \epsilon]$ with $M + \epsilon > M(u_0, r)$.

Lemma 4.4. *Suppose $r \ge 0$ and $u_0 \in H^r$. Then $M(u_0, r) = M(u_0, 0)$.*

Proof. By Lemma 4.1, it suffices to show that if $0 < M < M(u_0, 0)$ and u is a strong solution in $C([0,M], H^r)$ with initial data u_0, then $\|u(t)\|_{H^r}$ remains bounded for $t \in [0, M]$.

To see this, observe first that for all $t \in [0, M]$,

$$\begin{aligned}
\|u(t)\|_{H^r} &\le \|U(\alpha t)u_0\|_{H^r} + \int_0^t \|U(\alpha(t-t'))Q(u(t'))\|_{H^r}\ dt' \\
&= \|u_0\|_{H^r} + \int_0^t \|Q(u(t'))\|_{H^r}\ dt' \\
&\le \|u_0\|_{H^r} + \int_0^t \|u(t')\|_{L^2}^2 \|u(t')\|_{H^r}\ dt',
\end{aligned}$$

where we have used (3.13). Therefore

$$\|u(t)\|_{H^r} \le \|u_0\|_{H^r} + 3CR^2 \int_0^t \|u(t')\|_{H^r}\ dt',$$

where $R = \|u\|_{C([0,M],L^2)} < \infty$. Then from Gronwall's inequality it follows that, for all $t \in [0, M_1]$,

$$\|u(t)\|_{H^r} \le \|u_0\|_{H^r} e^{3CR^2 t} \le \|u_0\|_{H^r} e^{3CR^2 M_1} < \infty,$$

as desired. □

Lemma 4.5. *Suppose $u_0 \in H^2$. Then $M(u_0, 0) = \infty$, and $P(u(t))$ and $E(u(t))$ are independent of t for $t \ge 0$.*

Proof. Suppose $M < M(u_0, 0)$, so that a strong solution u with initial value u_0 exists in $C([0, M], L^2)$. By Lemma 4.4, u is also a strong solution in $C([0, M], H^2)$, and by Proposition 2.2, we have that $u \in C^1([0, M], L^2)$ as well.

In particular, it follows that $\frac{d}{dt}P(t) = \langle u_t, u\rangle + \langle u, u_t\rangle$ for $t \in [0, M]$. Now taking the inner product of (2.7) with u, and subtracting the resulting equation from its complex conjugate, we obtain that

$$-i\frac{d}{dt}P(u(t)) = \alpha\left(\langle u_{xx}, u\rangle - \langle u, u_{xx}\rangle\right) + \langle Q(u), u\rangle - \langle u, Q(u)\rangle. \tag{4.8}$$

for all $t \in [0, M]$. But for all $u \in H^2$, we have that $\langle u_{xx}, u\rangle = -\langle u_x, u_x\rangle$, which is a real quantity; and for all $u \in L^2$, we have that $\langle Q(u), u\rangle = \int_0^1 |T(s)u|^4\ ds$, which is also real. Therefore, we obtain from (4.8) that $\frac{d}{dt}P(u(t)) = 0$.

On the other hand, taking the inner product of (2.7) with u_t and adding the result to its complex conjugate yields

$$0 = -\alpha\left(\langle u_{xx}, u_t\rangle + \langle u_t, u_{xx}\rangle\right) - \left(\langle Q(u), u_t\rangle + \langle Q(u), u_t\rangle\right) = \frac{d}{dt}E(u(t)).$$

We have shown that $P(u(t))$ and $E(u(t))$ are constant for $t \in [0, M]$, for all $M < M(u_0, 0)$. Therefore $\|u(t)\|_{L^2}^2 = P(u(t))$ is constant for $0 \le t < M(u_0, 0)$, and as remarked before Lemma 4.4, this is enough to show that $M(u_0, 0) = \infty$. □

Lemma 4.6. *Suppose $u_0 \in L^2$. Then $M(u_0, 0) = \infty$, and $P(u(t))$ is independent of t for $t \ge 0$. If $\alpha = 0$, then $E(u(t))$ is independent of t for $t \ge 0$.*

Proof. Choose $K > 0$ such that $K > \|u_0\|_{L^2}$, and let M_K be the value of M asserted to exist in Lemmas 4.1 and 4.2 for $r = 0$ and this value of K, so that whenever $u_0, v_0 \in L^2$ with $\|u_0\|_{L^2} \leq K$ and $\|v_0\|_{L^2} \leq K$, the corresponding strong solutions u and v in $C([0, M_K], L^2)$ satisfy

$$\|u - v\|_{C([0,M_K],L^2)} \leq 2\|u_0 - v_0\|_{L^2}. \tag{4.9}$$

Let ϕ_n be a sequence of functions in H^2 such that $\|\phi_n\|_{L^2} \leq K$ for all n, and $\phi_n \to u_0$ in L^2. By Lemma 4.5, for each n there exists a strong solution v_n in $C([0, M_K], L^2)$ with $P(v_n(t)) = P(\phi_n)$ and $E(v_n(t)) = E(\phi_n)$ for all $t \in [0, M_K]$. From (4.9) we have that $v_n \to u$ in $C([0, M_K], L^2)$. Hence for all $t \in [0, M_K]$ we have $P(u(t)) = \lim_{n\to\infty} P(v_n(t)) = \lim_{n\to\infty} P(\phi_n) = P(u_0)$; and, if $\alpha = 0$, we also have using Lemma 3.8 that $E(u(t)) = \lim_{n\to\infty} E(v_n(t)) = \lim_{n\to\infty} E(\phi_n) = E(u_0)$.

Now, since we have that $\|u(M_K)\|_{L^2} = \|u_0\|_{L^2} < K$, we can repeat the argument with $u(M_K)$ as initial data, to obtain a strong solution $u \in C([0, 2M_K], L^2)$ with $P(u(t))$ constant for $t \in [0, 2M_K]$. Iterating this argument gives that $M(u_0, 0) = \infty$ and $P(u(t))$ is constant for all $t \geq 0$; moreover, $E(u(t))$ is constant for $t \geq 0$ if $\alpha = 0$. $\square$

Lemma 4.7. *For $r \geq 0$, the map from initial data to strong solutions in H^r is locally Lipschitz: for every $K > 0$ and $M > 0$, there exists $C > 0$ such that if $u_0, v_0 \in H^r$ with $\|u_0\|_{H^r} \leq K$ and $\|v_0\|_{H^r} \leq K$, and u and v are strong solutions in $C([0, M], H^r)$ with initial data u_0 and v_0, then*

$$\|u - v\|_{C([0,M],H^r)} \leq C\|u_0 - v_0\|_{H^r}.$$

Proof. Suppose $\|u_0\|_{H^r} \leq K$ and $\|v_0\|_{H^r} \leq K$, let M be given, and let u and v be the corresponding strong solutions in $C([0, M], H^r)$. From Lemma 4.6 we have that $\|u(t)\|_{L^2} \leq K$ and $\|v(t)\|_{L^2} \leq K$ for all t. Define $R = \max(\|u\|_{C([0,M],H^r)}, \|u\|_{C([0,M],H^r)})$. Then for all $t \in [0, M]$, we have from (3.14) that

$$\begin{aligned}
\|u(t) - v(t)\|_{H^r} &= \\
&= \left\| U(\alpha t)(u_0 - v_0) + \int_0^t U(\alpha(t - t'))\left[Q(u(t')) - Q(v(t'))\right] \, dt' \right\|_{H^r} \\
&\leq \|u_0 - v_0\|_{H^r} + \int_0^t \|Q(u(t')) - Q(v(t'))\|_{H^r} \, dt' \\
&\leq \|u_0 - v_0\|_{H^r} + CKR \int_0^t \|u(t') - v(t')\|_{H^r} \, dt'.
\end{aligned}$$

So from Gronwall's inequality it follows that

$$\|u(t) - v(t)\|_{H^r} \leq e^{CKRt}\|u_0 - v_0\|_{H^r} \leq e^{CKRM}\|u_0 - v_0\|_{H^r}$$

for all $t \in [0, M]$. $\square$

Lemma 4.8. *Suppose $u_0 \in L^2$ and $M > 0$. Then $\|u\|_{L^q_t([0,M],L^p_x)} < \infty$ for every admissible pair of exponents (q, p).*

Proof. From Lemma 3.2, and Lemma 3.3 with $(\gamma,\rho)=(\infty,2)$ and $(\gamma',\rho')=(1,2)$, we obtain the estimate

$$\begin{aligned}
&\|u\|_{L_t^q([0,M],L_x^p)} \le \\
&\quad \le \|U(\alpha t)u_0\|_{L_t^q([0,M],L_x^p)} + \left\| \int_0^t U(\alpha(t-t'))[Q(u(t')]\ dt' \right\|_{L_t^q([0,M],L_x^p)} \\
&\quad \le C\|u_0\|_{L^2} + C\|Q(u(t))\|_{L_t^1([0,M],L_x^2)}.
\end{aligned}$$

But, by (1.5),

$$\begin{aligned}
\|Q(u(t))\|_{L_t^1([0,M],L_x^2)} &= \int_0^M \|Q(u(t))\|_{L^2}\ dt \\
&\le C\int_0^M \|u(t)\|_{L^2}^3\ dt = CM\|u_0\|_{L^2}^3.
\end{aligned}$$

Therefore

$$\|u\|_{L_t^q([0,M],L_x^p)} \le C\left(\|u_0\|_{L^2} + M\|u_0\|_{L^2}^3\right) < \infty,$$

as desired. □

Taking Lemmas 4.1 through 4.8 together, we see that we have completed the proof of Theorem 2.3.

To prove Theorem 2.4, it suffices to make the following observations. First, the statement of Lemma 4.1 holds without change for equation (1.2), and the proof is the same as before, except Q is now replaced by Q_2, and in place of Lemmas 3.7 and 3.8, we use Lemma 3.9. The statements and proofs of Lemmas 4.2 and 4.3 remain the same for solutions of (1.2). Lemmas 4.4 through Lemma 4.8 follow for solutions of (1.2), with the same proofs as before, except that again we now use Lemma 3.9, and we replace Q by Q_2 and E by E_2 throughout.

5 Ill-posedness for 1DCR

In this section we prove Theorem 2.6. We assume throughout that $r<0$, $M>0$, and $\delta>0$ are given.

We will take

$$u_{0j}(x) = \beta\omega_j e^{iNx}\varphi(\omega_j x),$$

where $\varphi(x)$ is as defined in (2.12), and

$$\begin{aligned}
\beta &= N^{-r-(1/4)}, \\
\omega_1 &= \sqrt{N}, \\
\omega_2 &= \sqrt{N}(1+\delta),
\end{aligned} \tag{5.1}$$

and $N>0$ will be a large number to be chosen later. As seen in (2.10), a solution of (1.2) with this initial data is

$$u_j(x,t) = \beta\omega_j e^{i\beta^4\omega_j^2 t} e^{iNx}\varphi(\omega_j x).$$

Lemma 5.1. *For $j = 1$ or $j = 2$, define*

$$I_1(N) = \beta^2 \int_{|\xi| \le N/4} (1 + |\xi|^2)^r \left| \mathcal{F}\varphi \left(\frac{\xi - N}{\omega_j} \right) \right|^2 \, d\xi,$$
$$I_2(N) = \beta^2 \int_{|\xi| \ge 2N} (1 + |\xi|^2)^r \left| \mathcal{F}\varphi \left(\frac{\xi - N}{\omega_j} \right) \right|^2 \, d\xi.$$

Then $I_1(N) \to 0$ and $I_2(N) \to 0$ as $N \to \infty$.

Proof. For all ξ such that $|\xi| \ge 2N$, we have $|\xi - N| \ge |\xi|/2$, so

$$\frac{|\xi - N|}{\omega_j} \ge \frac{|\xi|}{2\sqrt{N}(1+\delta)} \ge \frac{\sqrt{N}}{1+\delta}. \tag{5.2}$$

Choose $C > 0$ such that

$$|\mathcal{F}\varphi(\eta)| \le C|\eta|^{-2} \quad \text{for } |\eta| \ge 1. \tag{5.3}$$

For N sufficiently large, it follows from (5.2) that $\frac{|\xi - N|}{\omega_j} \ge 1$, so (5.3) implies

$$\left| \mathcal{F}\varphi \left(\frac{|\xi - N|}{\omega_j} \right) \right| \le C \left| \frac{\xi - N}{\omega_j} \right|^{-2}.$$

Therefore using (5.2) we can write

$$\begin{aligned} I_2(N) &\le C\beta^2 \int_{|\xi| \ge 2N} (1 + |\xi|^2)^r \left| \frac{\xi - N}{\omega_j} \right|^{-2} \, d\xi \\ &\le CN^{-2r-(1/2)} \int_{|\xi| \ge 2N} |\xi|^{2r} \left(\frac{|\xi|}{2\sqrt{N}(1+\delta)} \right)^{-2} \, d\xi = C_1 N^{-1/2}, \end{aligned}$$

which proves that $I_2(N) \to 0$ as $N \to \infty$.

Also, for all ξ such that $|\xi| \le N/4$, we have $|\xi - N| \ge N/4$, so

$$\frac{|\xi - N|}{\omega_j} \ge \frac{\sqrt{N}}{4(1+\delta)},$$

which in particular is greater than 1 for N large. Therefore, by (5.3), we have

$$\begin{aligned} I_1(N) &\le C\beta^2 \int_{|\xi| \le N/4} (1 + |\xi|^2)^r \left| \frac{\xi - N}{\omega_j} \right|^{-2} \, d\xi \\ &\le C_1 N^{-2r-(1/2)} \int_{|\xi| \le N/4} N^{2r} N^{-1} \, d\xi = C_2 N^{-1/2}, \end{aligned}$$

and hence $I_1(N) \to 0$ as $N \to \infty$. $\square$

Lemma 5.2. *Suppose g is a continuously differentiable function on $\mathbf{R}$. Then for all $\rho > 1$,*

$$\int_{-\infty}^{\infty} |g(x) - g(\rho x)|^2 \, dx \le (\rho - 1)^2 \int_{-\infty}^{\infty} |x||g'(x)|^2 \, dx.$$

Proof. The desired estimate is established by the following computation, which is justified by Hölder's inequality and Tonelli's theorem:

$$\begin{aligned}
\int_{-\infty}^{\infty} |g(x) - g(\rho x)|^2 \, dx &= \int_{-\infty}^{\infty} \left| \int_x^{\rho x} g'(\xi) \, d\xi \right|^2 \, dx \\
&\leq \int_{-\infty}^{\infty} \left| \int_x^{\rho x} \, d\xi \right| \left| \int_x^{\rho x} g'(\xi)^2 \, d\xi \right| \, dx \\
&= (\rho - 1) \int_{-\infty}^{\infty} g'(\xi)^2 \left| \int_{\xi/\rho}^{\xi} x \, dx \right| \, d\xi \\
&= \frac{(\rho-1)^2}{\rho} \int_{-\infty}^{\infty} |\xi| |g'(\xi)|^2 \, d\xi.
\end{aligned}$$

□

Lemma 5.3. *For all $N \geq 0$, we have*

$$\beta^2 \int_{|\xi| \leq 2N} (1 + |\xi|^2)^r \left| \mathcal{F}\varphi\left(\frac{\xi - N}{\omega_1}\right) - \mathcal{F}\varphi\left(\frac{\xi - N}{\omega_2}\right) \right|^2 \, d\xi \leq C\delta^2$$

Proof. We have

$$\begin{aligned}
&\beta^2 \int_{|\xi| \leq 2N} (1 + |\xi|^2)^r \left| \mathcal{F}\varphi\left(\frac{\xi - N}{\omega_1}\right) - \mathcal{F}\varphi\left(\frac{\xi - N}{\omega_2}\right) \right|^2 \, d\xi \\
&\qquad \leq C\beta^2 N^{2r} \int_{|\xi| \leq 2N} \left| \mathcal{F}\varphi\left(\frac{\xi - N}{\omega_1}\right) - \mathcal{F}\varphi\left(\frac{\xi - N}{\omega_2}\right) \right|^2 \, d\xi \\
&\qquad \leq C\beta^2 N^{2r} \omega_1 \int_{-\infty}^{\infty} |\mathcal{F}\varphi(\xi) - \mathcal{F}\varphi(\rho\xi)|^2 \, d\xi \\
&\qquad \leq C(\rho - 1)^2 \int_{-\infty}^{\infty} |\xi| |(\mathcal{F}\varphi)'(\xi)|^2 \, d\xi,
\end{aligned}$$

where $\rho = \omega_1/\omega_2$, and we have used Lemma 5.2. Since

$$(\rho - 1)^2 = \frac{\delta^2}{(1+\delta)^2} \leq \delta^2,$$

and $\int_{-\infty}^{\infty} |\xi| |(\mathcal{F}\varphi)'(\xi)|^2 \, d\xi < \infty$, this proves the lemma. □

Lemma 5.4. *We have*

$$\|u_{01} - u_{02}\|_{H^r} \leq C\delta + o(N) \tag{5.4}$$

as $N \to \infty$, for some C which is independent of N.

Proof. Writing

$$\begin{aligned}
\|u_{01} - u_{02}\|_{H^r}^2 \leq \beta^2 &\int_{|\xi| \geq 2N} (1 + |\xi|^2)^r \left| \mathcal{F}\varphi\left(\frac{\xi - N}{\omega_1}\right) \right|^2 \, d\xi \\
&+ \beta^2 \int_{|\xi| \geq 2N} (1 + |\xi|^2)^r \left| \mathcal{F}\varphi\left(\frac{\xi - N}{\omega_2}\right) \right|^2 \, d\xi \\
&+ \beta^2 \int_{|\xi| \leq 2N} (1 + |\xi|^2)^r \left| \mathcal{F}\varphi\left(\frac{\xi - N}{\omega_1}\right) - \mathcal{F}\varphi\left(\frac{\xi - N}{\omega_2}\right) \right|^2 \, d\xi,
\end{aligned}$$

we see that the desired estimate (5.4) follows from Lemmas 5.1 and 5.3. □

Lemma 5.5. *For all sufficiently large N, we have*

$$2^{2r-1}\|\varphi\|_{L^2} \le \|u_{0j}\|_{H^r} \le 2\|\varphi\|_{L^2}$$

for $j = 1, 2$.

Proof. Let j be either 1 or 2. We write

$$\|u_{0j}\|^2_{H^r} = I_1(N) + I_2(N) + I_3(N),$$

where

$$I_m(N) = \beta^2 \int_{E_m} (1+|\xi|^2)^r \left|\mathcal{F}\varphi\left(\frac{\xi - N}{\omega_j}\right)\right|^2 \, d\xi$$

for $m = 1, 2, 3$, with

$$\begin{aligned} E_1 &= \{|\xi| \le N/4\}\,, \\ E_2 &= \{|\xi| \ge 2N\}\,, \\ E_3 &= \{N/4 \le |\xi| \le 2N\}\,. \end{aligned}$$

By Lemma 5.1, we have $I_1(N) = o(N)$ and $I_2(N) = o(N)$ as $N \to \infty$. Also we have

$$I_3(N) \le \beta^2 N^{2r} \int_{\{N/4 \le |\xi| \le 2N\}} \left|\mathcal{F}\varphi\left(\frac{\xi - N}{\omega_j}\right)\right|^2 \, d\xi \le 2^{-2r} I_3(N),$$

and

$$\begin{aligned} &\beta^2 N^{2r} \int_{\{N/4 \le |\xi| \le 2N\}} \left|\mathcal{F}\varphi\left(\frac{\xi - N}{\omega_j}\right)\right|^2 \, d\xi = \\ &= N^{-1/2} \int_{-\infty}^{\infty} \left|\mathcal{F}\varphi\left(\frac{\xi - N}{\omega_j}\right)\right|^2 \, d\xi + o(N) = \\ &= N^{-1/2}\omega_j \|\mathcal{F}\varphi\|^2_{L^2} + o(N) = N^{-1/2}\omega_j\|\varphi\|^2_{L^2} + o(N). \end{aligned}$$

Since $N^{-1/2}\omega_1 = 1$ and $N^{-1/2}\omega_2 = (1 + \delta)$, these estimates are enough to complete the proof. □

Lemma 5.6. *Let $M > 0$ be given. Then there exists a constant C, independent of δ, with the following property: for every $\delta > 0$ one can find a sequence N_k tending to infinity such that for $N = N_k$ one has*

$$\|u_1(x, M) - u_2(x, M)\|_{H^r} \ge 2^{r-1}\|\varphi\|_{L^2} - C\delta. \tag{5.5}$$

Proof. We have

$$
\begin{aligned}
&\|u_1(x,M)-u_2(x,M)\|_{H^r}^2 = \\
&= \beta^2 \int_{-\infty}^{\infty} (1+|\xi|^2)^r \left| e^{iM\omega_1^2}\mathcal{F}\varphi\left(\frac{\xi-N}{\omega_1}\right) - e^{iM\omega_2^2}\mathcal{F}\varphi\left(\frac{\xi-N}{\omega_2}\right)\right|^2 \, d\xi \\
&= \beta^2 \int_{-\infty}^{\infty} (1+|\xi|^2)^r \left| \mathcal{F}\varphi\left(\frac{\xi-N}{\omega_1}\right) - e^{iM(\omega_2^2-\omega_1^2)}\mathcal{F}\varphi\left(\frac{\xi-N}{\omega_2}\right)\right|^2 \, d\xi \\
&\geq \left|1-e^{iM(\omega_2^2-\omega_1^2)}\right|^2 \beta^2 \int_{-\infty}^{\infty} (1+|\xi|^2)^r \left| \mathcal{F}\varphi\left(\frac{\xi-N}{\omega_2}\right)\right|^2 \, d\xi \\
&\quad - \beta^2 \int_{-\infty}^{\infty} (1+|\xi|^2)^r \left| \mathcal{F}\varphi\left(\frac{\xi-N}{\omega_1}\right) - \mathcal{F}\varphi\left(\frac{\xi-N}{\omega_2}\right)\right|^2 \, d\xi \\
&= \left|1-e^{iM(\omega_2^2-\omega_1^2)}\right|^2 \|u_{02}\|_{H^r}^2 - \|u_{01}-u_{02}\|_{H^r}^2.
\end{aligned}
$$

By Lemmas 5.4 and 5.5, it follows that

$$
\|u_1(x,M)-u_2(x,M)\|_{H^r} \geq 2^{r-1}\|\varphi\|_{L^2}\left|1-e^{iM(\omega_2^2-\omega_1^2)}\right| - C\delta \tag{5.6}
$$

for N sufficiently large.

Now observe that

$$
\left|1-e^{iM(\omega_2^2-\omega_1^2)}\right| = \left|1-e^{iMN^{-4r}\delta(2+\delta)}\right|.
$$

Therefore if we choose

$$
N = N_k = \left(\frac{\pi(2k+1)}{M\delta(2+\delta)}\right)^{-1/4r},
$$

we will have

$$
\left|1-e^{iM(\omega_2^2-\omega_1^2)}\right| = 1,
$$

and hence (5.5) follows from (5.6). ☐

Proof of Theorem 2.6. For a given $\delta > 0$, choose $\delta_1 > 0$ such that $C\delta_1 < \delta/2$, where C is the constant in Lemma 5.4, and $C\delta < 2^{r-2}\|\varphi\|_{L^2}$, where C is the constant in Lemma 5.6. Replace δ by δ_1 in (5.1) and the lemmas which follow. Then from Lemmas 5.4 and 5.5 it follows that (2.16) and (2.17) hold, with $u = u_1$ and $v = u_2$, for sufficiently large N, say for $N \geq N_0$. On the other hand, by Lemma 5.6 we can find $N \geq N_0$ for which (5.5) holds as well, and hence (2.18) also holds for $u = u_1$ and $v = u_2$. This completes the proof.

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