ON THE EXACT SOLUTIONS OF THE INTERMEDIATE LONG-WAVE EQUATION

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To the memory of Peter Hess

1. Introduction. The intermediate long-wave equation was introduced by R. I. Joseph [4] as a mathematical model of nonlinear dispersive waves on the interface between two fluids of different positive densities contained at rest in a long channel with a horizontal top and bottom, the lighter fluid forming a horizontal layer above a layer of the same depth of the heavier fluid. When variables have been re-scaled it is the pseudo-differential operator equation (see [5])

$$\eta_t + 2\eta\eta_x - (N_H\eta)_x + (1/H)\eta_x = 0, \tag{1}$$

where $H > 0$ and the Fourier multiplier operator N_H is given by

$$\widehat{N_H\eta}(k) = (k \coth kH)\widehat{\eta}(k).$$

In common with the classical KdV and Benjamin-Ono equations, between which it was intended to form a model-theoretical bridge [4], equation (1) was found to have a family of exact solitary-wave solutions: namely,

$$\eta(x,t) = \phi_{C,H}(x - Ct),$$

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where

$$\phi_{C,H}(x) = \left[\frac{a \sin aH}{\cosh ax + \cos aH}\right], \qquad x \in \mathbb{R},$$

for arbitrary $C > 0$ and $H > 0$, and a is the unique solution of the transcendental equation

$$aH \cot aH = (1 - CH), \qquad a \in (0, \pi/H).$$

Hence $\phi_{C,H}$ is an L^2 solution of the steady travelling wave equation

$$N_H\psi + \mu\psi = \psi^2, \qquad \mu = C - (1/H). \tag{2}$$

(For general L^2 functions ψ, both sides of (2) are well-defined tempered distributions. We will see below that any L^2 solution of (2) is a C^∞ function and hence yields a classical solution of (1).) Explicit dependence of (2) upon H may be scaled away by putting

$$\phi(x) = H\psi(Hx) \quad \text{and} \quad \widehat{N\phi}(k) = (k \coth k)\widehat{\phi}(k).$$

Then

$$(N + \gamma)\phi = \phi^2, \qquad \gamma > -1. \tag{3}$$

A study of the set of all L^2 solutions for $\gamma > -1$ of (3) is therefore equivalent to a study of the set of all L^2 solutions of (2) for all positive values of H and C.

The rest of this paper focuses on (3), and we prove the following theorem. Let $\gamma > -1$ be fixed and let

$$\sigma \cot \sigma + \gamma = 0, \qquad \sigma \in (0, \pi).$$

Theorem. *Suppose that ϕ is an L^2 solution of (3). Then*

(a)

$$\left(\frac{d\phi}{dx}(x)\right)^2 = 2\gamma\phi^3(x) - \phi^4(x) + \sigma^2\phi^2(x),\ x \in \mathbb{R};$$

(b) *for some $p \in \mathbb{R}$,*

$$\phi(x + p) = \mathit{Real}\ f(x + i0),$$

where f is a solution on $\mathbb{R} \times [0,1] \subset \mathbb{C}$ of the complex analytic initial-value problem

$$\frac{\partial f}{\partial z} = \frac{i}{2}\left[f(z)^2 + \sigma^2\right],\ f(0,0) = \phi(p);$$

(c) *If u is a harmonic function in the strip $\mathbb{R} \times [0,1]$ with $u(x,0) = \phi(x)$ and $u(x,1) = 0$, then*

$$\frac{\partial u}{\partial y} = \gamma u - u^2 \quad on \quad y = 0.$$

(d) *For $(x,y) \in \mathbb{R} \times [0,1]$,*

$$\frac{\partial u}{\partial y}(x,y) = \gamma(y)u(x,y) - u^2(x,y)$$

and

$$u_{xx}(x,y) - 3\gamma(y)u^2(x,y) + 2u^3(x,y) - \sigma^2 u(x,y) = 0,$$

where

$$\gamma(y) = \sigma \cot(\sigma(y-1)).$$

Remarks.

1. The ordinary differential equation in part (a) is easily solved explicitly to yield the uniqueness (up to translation) of the known solitary-wave solution of the intermediate long-wave equation

$$\phi(x) = \left[\frac{\sigma \sin \sigma}{\cosh \sigma x + \cos \sigma}\right].$$

2. The key to the proof is part (c), which follows easily by the Fourier inversion formula. Part (b) then follows by the maximum principle as in the treatment of the Benjamin-Ono equation in [3]. (As was the case in [3], the basic maximum principle for harmonic functions is not enough for our purposes: rather, the proof relies heavily on the maximum principle and Hopf boundary point lemma for general elliptic inequalities [6].) From a pedagogical viewpoint it is worth emphasizing that every solution of the

complex equation in (b) has a real part which satisfies part (d) away from its poles. It is routine to check that the real part then yields a solution of the intermediate long wave equation on $\{y = 0\}$. In this sense, the function f might be regarded as a kind of generating function for the function u in (c) and ϕ in equation (3).

3. It is possible to derive part (a) of the Theorem as a consequence of the properties of the operator N on $L^2(\mathbb{R})$, without appealing to maximum principle arguments in the plane. This approach is detailed in [1].

Notation. Throughout we use $L^p(\mathbb{R})$ and $H^s(\mathbb{R})$ to denote the usual Lebesgue and Sobolev spaces of (equivalence classes of) functions on $\mathbb{R}$. Recall that every function in $H^1(\mathbb{R})$ is continuous. If $f \in H^1(\mathbb{R})$ then $\widehat{f}$ is in $L^1(\mathbb{R})$, where $\widehat{f}$, the Fourier transform of f, is defined by

$$\widehat{f}(k) = \int_{-\infty}^{\infty} e^{ikx} f(x)\, dx.$$

2. Preliminaries.

The proof will involve studying a solution u of Laplace's equation on the infinite strip $S = \mathbb{R} \times (0, 1)$ in $\mathbb{R}^2$, with Dirichlet data equal to ϕ on the lower boundary of the strip. In order to make use of maximum principle arguments to study u, it will be necessary to establish that u is regular on S up to the boundary of S. The next two Lemmas show that the desired regularity properties of u can be deduced from a priori regularity estimates on ϕ.

Lemma 1. *If ϕ is any L^2 solution of (3) then ϕ, $N\phi$ and all their derivatives are in L^p for $1 \leq p \leq \infty$.*

Proof. If (3) holds, then the Fourier transform of ϕ satisfies the equation $\widehat{\phi}(k) = (k \coth k + \gamma)^{-1} \widehat{\phi^2}(k)$; which may be rewritten in convolution form as

$$\phi = K * \phi^2 \tag{4}$$

where $\widehat{K}(k) = (k \coth k + \gamma)^{-1}$.

Now we claim that $K \in L^p$ for $1 \le p < \infty$. To see this, first write the even function K as the inverse Fourier transform $K(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} [k\coth k + \gamma]^{-1} e^{-ikx}\, dk$, and then use Jordan's Lemma and the Residue Theorem to arrive at the formula

$$K(x) = \sum_{j=0}^{\infty} e^{-\theta_j |x|}\left[\frac{2\sin^2\theta_j}{2\theta_j - \sin(2\theta_j)}\right]$$

where $\{\theta_j\}_{j=0,1,2,\ldots}$ are the positive solutions of $\theta_j \cot\theta_j + \gamma = 0$. Since $j\pi < \theta_j < (j+1)\pi$ for all $j \ge 0$, one can find a constant A (independent of x) such that

$$\begin{aligned} |K(x)| &\le A\left[e^{-\theta_0|x|} + \sum_{j=1}^{\infty}\left(\frac{1}{j}\right)e^{-j\pi|x|}\right] \\ &= A\left[e^{-\theta_0|x|} + |\log\left(1 - e^{-\pi|x|}\right)|\right]. \end{aligned}$$

But the latter function decays exponentially as $|x| \to \infty$ and blows up only logarithmically as $x \to 0$. Thus the claim is proved.

Next it will be proved, by induction on j, that $(\frac{d}{dx})^j\phi \in L^p$ for $1 \le p < \infty$. The statement for $j = 0$ follows immediately from the formula $\phi = K * \phi^2$, the fact that $\phi^2 \in L^1$, the just-proved claim, and Young's convolution inequality. Assume now that the statement has been proved for $0, 1, 2, \ldots, j$. Then the derivatives of ϕ of order up to j are in L^4, and it follows from Holder's inequality that $(\frac{d}{dx})^j(\phi^2) \in L^2$. But the operator $(\frac{d}{dx})(N+\gamma)^{-1}$ is bounded on L^2, as it is a Fourier multiplier operator with the bounded multiplier $(-ik)[k\coth k + \gamma]^{-1}$. Since $(\frac{d}{dx})^{j+1}\phi = \frac{d}{dx}(N+\gamma)^{-1}(\frac{d}{dx})^j(\phi^2)$ by (3), then $(\frac{d}{dx})^{j+1}\phi \in L^2$ also. Holder's inequality and the induction hypothesis now imply that $(\frac{d}{dx})^{j+1}(\phi^2) \in L^1$. From $(\frac{d}{dx})^{j+1}\phi = K * (\frac{d}{dx})^{j+1}(\phi^2)$ it now follows that $(\frac{d}{dx})^{j+1}\phi \in L^p$ for all $1 \le p < \infty$, and so the inductive proof is complete.

Since $L^\infty(\mathbb{R})$ is contained in $H^1(\mathbb{R})$, a consequence of what has just been proved is that all the derivatives of ϕ are also in L^∞.

Finally, since $N\phi = \phi^2 - \gamma\phi$ by (3), it follows easily that $N\phi$ and all its derivatives are in L^p for $1 \le p \le \infty$. $\square$

Lemma 2. *Let ϕ be an L^2 solution of (3). There exists a function $u \in C^2(\bar{S})$ such that*

(i) $\Delta u = 0$ *on* S

(ii) $u(x,0) = \phi(x)$ *for* $x \in \mathbb{R}$

(iii) $u(x,1) = 0$ *for* $x \in \mathbb{R}$

(iv) $u(x,y) \to 0$, *uniformly for* $y \in [0,1]$, *as* $|x| \to \infty$

(v) $u_x(x,y) \to 0$, *uniformly for* $y \in [0,1]$, *as* $|x| \to \infty$

(vi) $u_y(x,0) = -N\phi(x)$ *for* $x \in \mathbb{R}$. *In particular,*

$$u_y(x,0) = \gamma u(x,0) - u^2(x,0), \qquad x \in \mathbb{R}.$$

Proof. Once it is known that ϕ is sufficiently regular, this result is a consequence of standard potential theory. For completeness, however, we give the simple, self-contained proof here.

For $(x,y) \in \bar{S}$ define

$$u(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \left[\frac{\sinh k(1-y)}{\sinh k} \right] \widehat{\phi}(k)\ dk.$$

By Lemma 1, $\widehat{\phi}$ is bounded and integrable. So for $y > 0$ the rapid decay of the function in brackets in the integrand allows differentiation under the integral as often as desired and property (i) follows. Property (iii) is immediate from the formula and property (ii) is a consequence of the Fourier inversion formula.

To prove properties (iv) and (v), observe that for $y > 0$ the formula for $u(x,y)$ may be rewritten as

$$u(x,y) = \int_{-\infty}^{\infty} H(y;z)\phi(x-z)\ dz$$

where

$$\widehat{H(y;\cdot)}(k) = \frac{\sinh k(1-y)}{\sinh k}.$$

A calculation (or a table of Fourier transforms) shows that

$$H(y;x) = \frac{(1/2)\sin\delta}{[\cosh(\pi x) + \cos\delta]}, \qquad \delta = \pi(1-y)$$

and clearly

$$\sup_{y \in [0,1]} \int_{|z| \geq R} H(y;z)\ dz \to 0 \quad \text{as } R \to \infty. \tag{4}$$

It follows that for all $x \in \mathbb{R}$

$$
\begin{aligned}
\sup_{y\in[0,1]} |u(x,y)| \le & \left(\sup_{y\in[0,1]} \int_{|z|\le|x|/2} H(y;z)|\phi(x-z)|\ dz \right) \\
& \qquad + \left(\sup_{y\in[0,1]} \int_{|z|\ge|x|/2} H(y;z)|\phi(x-z)|\ dz \right) \\
\le & \left(\sup_{y\in[0,1]} \int_{-\infty}^{\infty} H(y;z)\ dz \right) \left(\sup_{|w|\ge|x|/2} |\phi(w)| \right) \\
& \qquad + \left(\sup_{y\in[0,1]} \int_{|z|\ge|x|/2} H(y;z)\ dz \right) \left(\sup_{w\in\mathbb{R}} |\phi(w)| \right).
\end{aligned}
$$

Since $\phi \in H^1$, then $\phi(x) \to 0$ as $|x| \to \infty$, and so the first term in the last expression tends to zero as $|x| \to \infty$. The second term also tends to zero, by (4). Thus (iv) has been proved. The proof of (v) proceeds similarly from the fact that $u_x(x,y) = \int_{-\infty}^{\infty} H(y;z)\phi'(x-z)\ dz$.

Property (vi) follows immediately from the definition of u by differentiation under the integral with respect to y, which is justified by the Dominated Convergence Theorem and the fact that $k\widehat{\phi}(k)$ is an L^1 function of k.

It remains only to show that u is C^2 at the boundary $y = 0$. For this, it suffices to prove that u and its partial derivatives up to second order converge uniformly, as functions of x, to their boundary values as $y \to 0$. But for all $x \in \mathbb{R}$,

$$
\begin{aligned}
|u(x,y) - u(x,0)| &= \left| \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \left[\left(\frac{\sinh k(1-y)}{\sinh k} \right) - 1 \right] \widehat{\phi}(k)\ dk \right| \\
&\le \frac{1}{2\pi} \int_{-\infty}^{\infty} \left| \left(\frac{\sinh k(1-y)}{\sinh k} \right) - 1 \right| |\widehat{\phi}(k)|\ dk,
\end{aligned}
$$

and the convergence of the latter integral to zero as $y \to 0$ is assured by the Dominated Convergence Theorem. Similar arguments prove that $u_x(x,y) \to u_x(x,0)$ and $u_{xx}(x,y) \to u_{xx}(x,0)$ uniformly as $y \to 0$. Also, using (vi), one has

$$
|u_y(x,y) - u_y(x,0)| = \left| \frac{-1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \left[\frac{k\cosh k(1-y)}{\sinh k} - k\coth k \right] \widehat{\phi}(k)\ dk \right|,
$$

and so again uniform convergence as $y \to 0$ follows from the Dominated Convergence Theorem. Finally, since $u_{yy} = -u_{xx}$ for $y > 0$, it follows that u_{yy} also converges, uniformly in x, as $y \to 0$. This completes the proof of the Lemma. □

Remark. It follows from the maximum principle for harmonic functions, applied to domains $[-R, R] \times [0, 1]$, for R sufficiently large, that the conditions (i)–(iv) of Lemma 2 determine u uniquely in $C^2(\bar{S})$.

3. Proof of the main theorem.

The proof of the main theorem is now done via a sequence of lemmas. Since the theorem is obviously true when the solution ϕ of (3) is identically zero, it will be assumed in what follows that ϕ is non-trivial (and hence that the function u defined in Lemma 2 is not identically zero on S). We begin with some estimates.

Lemma 3. *The function u defined in Lemma 2 satisfies*

(i) $u(x, y) > 0$ *for all* $(x, y) \in \mathbb{R} \times [0, 1)$

(ii) $u_y(x, 1) < 0$ *for all* $x \in \mathbb{R}$.

Proof. First it will be shown that $u \geq 0$ on $\bar{S}$. Recall that $\gamma > -1$ and suppose that u takes a negative value at some point of $\bar{S}$. Then the function $w(x, y) = \dfrac{u(x, y)}{1 + \gamma y}$ also takes a negative value at some point of $\bar{S}$. Moreover,

$$w \to 0 \quad \text{(uniformly for } y \in [0, 1]) \quad \text{as} \quad |x| \to \infty,$$

and w satisfies the elliptic equation

$$\Delta w + \left(\frac{2\gamma}{1 + \gamma y} \right) w_y = 0.$$

Therefore, w satisfies a maximum principle (cf. Theorem 5, p. 61 of [6]) which implies that the minimum value of w must be attained at a point on the boundary of $\bar{S}$. Since w is zero on the line $y = 1$, then the minimum value of w is attained at a point where $y = 0$. Moreover, by the Hopf boundary-point lemma (Theorem 7, p. 65 of [6]), $w_y > 0$ at this point. But when $y = 0$, one has $w_y = u_y - \gamma u$, and $u_y = \gamma u - u^2$ by Lemma 2(ii), (vi), and equation (3). Together these equations imply that $w_y = -u^2 \leq 0$ for $y = 0$, which is a contradiction. Thus $u \geq 0$ on $\bar{S}$.

Since u is not identically zero on S, it now follows from the maximum principle that $u > 0$ at all points in S. To prove that $u > 0$ also holds on $\{y = 0\}$, one assumes that $u = 0$ at some point where $y = 0$, and uses the fact that w has a minimum value at that point to derive a contradiction by the Hopf boundary point lemma, as in the preceding paragraph. Thus the proof of (i) is complete.

To prove (ii), observe that from (i) and Lemma 2(iii) it follows that u attains its minimum value on $\bar{S}$ at all the points where $y = 1$. That $u_y(x, 1) < 0$ for all x is immediate by the Hopf boundary point lemma. □

The next four Lemmas contain estimates of the behavior of u and u_x as $|x| \to \infty$. These estimates are obtained by comparing u and u_x to functions of the form

$$\Gamma_\alpha(x, y) = e^{-(\frac{\pi}{2}+\alpha)x} \cos\left(\left(\frac{\pi}{2} + \alpha\right) y - \alpha\right), \qquad \alpha \in (-\pi/2, \pi/2).$$

These functions are harmonic, positive for $0 \le y < 1$, and satisfy

$$\Gamma_\alpha = 0,\ y = 1 \quad \text{and} \quad \frac{\partial(\Gamma_\alpha)}{\partial y} = \left[\left(\frac{\pi}{2} + \alpha\right) \tan \alpha\right] \Gamma_\alpha, \qquad y = 0.$$

In particular,

$$(\Gamma_{\tilde{\alpha}})_y = \gamma(\Gamma_{\tilde{\alpha}}) \quad \text{if} \quad y = 0 \quad \text{and} \quad (\pi/2 + \tilde{\alpha}) \tan \tilde{\alpha} = \gamma.$$

Lemma 4. *There exists $A > 0$ such that*

$$u(x, y) \ge A\Gamma_{\tilde{\alpha}}(x, y) \quad \textit{for all} \quad (x, y) \in [0, \infty) \times [0, 1],$$

and

$$u(x, y) \ge A\Gamma_{\tilde{\alpha}}(-x, y) \quad \textit{for all} \quad (x, y) \in (-\infty, 0] \times [0, 1].$$

Proof. By Lemma 3, $u(0, y) > 0$ for all $y \in [0, 1)$ and $u_y(0, 1) < 0$. Hence a number $A > 0$ can be chosen so that $u(0, y) \ge A\Gamma_{\tilde{\alpha}}(0, y)$ for all $y \in [0, 1)$. Let $\Omega = (0, \infty) \times (0, 1)$. Then the function

$$w(x, y) = \left(\frac{u(x, y) - A\Gamma_{\tilde{\alpha}}(x, y)}{1 + \gamma y}\right)$$

satisfies an elliptic equation on Ω, as in the proof of Lemma 3, and hence obeys a maximum principle. Moreover, w vanishes (uniformly in y) as $x \to \infty$; so that if w

takes a negative value in Ω then w must attain its negative minimum at some finite boundary point of Ω. Such a minimum point cannot occur on $\{y = 1\}$, where $w = 0$, or on $\{x = 0\}$, where $w \geq 0$, and hence must occur on the line $\{y = 0,\ x > 0\}$ at a point where $w_y > 0$. But, as in the proof of Lemma 3, for $y = 0$ one has

$$\begin{aligned} w_y &= u_y - \gamma u - ((\Gamma_{\tilde{\alpha}})_y - \gamma \Gamma_{\tilde{\alpha}}) \\ &= u_y - \gamma u = -u^2 \leq 0. \end{aligned}$$

This contradiction shows that $w \geq 0$ on Ω. The corresponding assertion on $(-\infty, 0] \times [0, 1]$ is proved similarly. □

Lemma 5. *For every $\alpha \in (-\pi/2, \tilde{\alpha})$, there exists $B_\alpha > 0$ such that*

$$u(x, y) \leq B_\alpha \Gamma_\alpha(x, y) \quad \text{for all} \quad (x, y) \in [0, \infty) \times [0, 1],$$

and

$$u(x, y) \leq B_\alpha \Gamma_\alpha(-x, y) \quad \text{for all} \quad (x, y) \in (-\infty, 0] \times [0, 1].$$

Proof. Let $\delta = (\frac{\pi}{2} + \alpha) \tan \alpha,\ \alpha \in (-\pi/2, \tilde{\alpha})$. Then $\gamma - \delta > 0$, so $X \in \mathbb{R}$ can be found such that $u(x, y) \leq \frac{1}{3}(\gamma - \delta)$ for all $x \geq X$. Choose $C > 0$ such that $u(X, y) \leq C\Gamma_\alpha(X, y)$ for all $y \in [0, 1]$. Let $\Omega = (X, \infty) \times (0, 1)$; then the function

$$w(x, y) = \left(\frac{u(x, y) - C\Gamma_\alpha(x, y)}{1 + \delta y} \right)$$

satisfies an elliptic equation on Ω and vanishes, uniformly in y, as $x \to \infty$. Moreover, w vanishes on the line $\{y = 1\}$ and is non-positive for $\{x = X\}$. Therefore it follows, as in the proof of the preceding Lemma, that if w takes a positive value on Ω then w must have a positive maximum value on the line $\{y = 0\}$, and $w_y < 0$ there. But for $y = 0$ and $x \geq X$ one finds that

$$\begin{aligned} w_y &= u_y - \delta u - C((\Gamma_\alpha)_y - \delta \Gamma_\alpha) \\ &= \gamma u - u^2 - \delta u = (\gamma - \delta - u)u > 0, \end{aligned}$$

by the choice of X. This contradiction shows that $w \leq 0$ on Ω, so $u \leq C\Gamma_\alpha$ on Ω. A similar argument shows the existence of a number $X' < 0$ and a constant $C' > 0$ such that $u(x, y) \leq C'\Gamma_\alpha(-x, y)$ for $x \leq X'$ and $y \in [0, 1]$. The statement of the Lemma now follows easily. □

Lemma 6. *For every* $\alpha \in (-\pi/2, \tilde{\alpha})$, *there exists* $D_\alpha > 0$ *such that*

$$u_x(x,y) \geq -D_\alpha \Gamma_\alpha(x,y) \quad \textit{for all} \quad (x,y) \in [0,\infty) \times [0,1],$$

and

$$u_x(x,y) \geq -D_\alpha \Gamma_\alpha(-x,y) \quad \textit{for all} \quad (x,y) \in (-\infty,0] \times [0,1].$$

Proof. Define δ, X, X', and Ω as in the proof of Lemma 5, and choose $D > 0$ such that

$$u_x(X,y) \geq -D\Gamma_\alpha(X,y) \quad \text{for all} \quad y \in [0,1].$$

Then the function

$$w(x,y) = \left(\frac{u_x(x,y) + D\Gamma_\alpha(x,y)}{1+\delta y} \right)$$

satisfies an elliptic equation on Ω, is zero on $\{y=1\}$, and tends to zero as $x \to \infty$ by Lemma 2(v). The same argument as in the proof of Lemma 5 then shows that if w takes a negative value on Ω, then it must attain a negative minimum at a point on the line $\{y=0\}$ where $w_y > 0$. But by Lemma 2,

$$\begin{aligned} w_y &= (u_{xy} - \delta u_x) + D((\Gamma_\alpha)_y - \delta \Gamma_\alpha) \\ &= (u_y - \gamma u)_x + (\gamma - \delta) u_x \\ &= (-2u + \gamma - \delta) u_x, \qquad y = 0. \end{aligned}$$

Now, at the point $(x_0, 0) \in \partial\Omega$ where w has its minimum, one has $u(x_0,0) < (\gamma-\delta)/2$ (since $x_0 > X$) and $u_x(x_0,0) < 0$ (otherwise $w(x_0,0)$ would be a positive number). Hence $w_y(x_0,0) < 0$. This contradiction shows that $u_x \geq -D\Gamma_\alpha$ on Ω. A similar argument shows the existence of $D' > 0$ such that $u_x(x,y) \geq -D'\Gamma_\alpha(-x,y)$ on $(-\infty, X'] \times [0,1]$, and the statement of the Lemma follows. □

Lemma 7. *There exists* $D_{\tilde{\alpha}} > 0$ *such that*

$$u_x(x,y) \geq -D_{\tilde{\alpha}} \Gamma_{\tilde{\alpha}}(x,y) \quad \textit{for all} \quad (x,y) \in [0,\infty) \times [0,1],$$

and

$$u_x(x,y) \geq -D_{\tilde{\alpha}} \Gamma_{\tilde{\alpha}}(-x,y) \quad \textit{for all} \quad (x,y) \in (-\infty,0] \times [0,1].$$

Proof. Since $\tilde{\alpha} \in (-\pi/2, \pi/2)$, a number $\alpha_1 \in (-\pi/2, \pi/2)$ can be found such that $\tilde{\alpha} < \alpha_1 < \frac{\pi}{2} + 2\tilde{\alpha}$. From the last inequality it then follows that a number $\alpha_2 \in (-\pi/2, \pi/2)$ may be chosen so that $\frac{\alpha_1 - (\pi/2)}{2} < \alpha_2 < \tilde{\alpha}$. In particular, the relation $2(\frac{\pi}{2} + \alpha_2) > \frac{\pi}{2} + \alpha_1$ holds.

Now by Lemmas 3, 5, and 6, there exist constants $B_{\alpha_2} > 0$ and $D_{\alpha_2} > 0$ such that $0 \leq u(x,y) \leq B_{\alpha_2}\Gamma_{\alpha_2}(x,y)$ and $u_x(x,y) \geq -D_{\alpha_2}\Gamma_{\alpha_2}(x,y)$ for all $(x,y) \in [0,\infty) \times [0,1]$. It follows that for all $(x,y) \in [0,\infty) \times [0,1]$,

$$\begin{aligned}\frac{uu_x}{\Gamma_{\alpha_1}} &\geq \frac{-B_{\alpha_2}D_{\alpha_2}(\Gamma_{\alpha_2}(x,y))^2}{\Gamma_{\alpha_1}(x,y)} \\ &= -B_{\alpha_2}D_{\alpha_2}\exp([-2(\pi/2+\alpha_2)+(\pi/2+\alpha_1)]x)\left(\frac{\sin^2((\pi/2+\alpha_2)(1-y))}{\sin((\pi/2+\alpha_1)(1-y))}\right).\end{aligned}$$

But the argument of the exponential in the right-hand side is negative, and the quotient of sine functions is a positive bounded function of y on $[0,1)$. Therefore $\left(\frac{uu_x}{\Gamma_{\alpha_1}}\right)$ becomes greater than any given negative number as $x \to \infty$ (uniformly for $y \in [0,1]$). In particular, if $\delta_1 = (\pi/2 + \alpha_1)\tan\alpha_1$, then $\gamma - \delta_1 < 0$, and so there exists $X \geq 0$ such that $\left(\frac{2uu_x}{\Gamma_{\alpha_1}}\right) \geq \gamma - \delta_1$ for all $(x,y) \in [X,\infty) \times [0,1]$.

We claim there exists $C > 0$ such that

$$u_x(x,y) - \Gamma_{\alpha_1}(x,y) \geq -C\Gamma_{\tilde{\alpha}}(x,y)$$

for all $(x,y) \in [X,\infty) \times [0,1]$. To see this, choose $C > 0$ such that the inequality holds for $x = X$ and all $y \in [0,1]$. Then the function

$$w(x,y) = \frac{u_x(x,y) - \Gamma_{\alpha_1}(x,y) + C\Gamma_{\tilde{\alpha}}(x,y)}{(1+\gamma y)}$$

satisfies an elliptic equation on $\Omega = (X,\infty) \times (0,1)$, vanishes as $x \to \infty$, and is non-negative on the boundary where $x = X$ or $y = 1$. Hence if w takes any negative value in Ω, it must attain a negative minimum at a point in the boundary of Ω on the line $\{y = 0\}$; and at that point one must have $w_y > 0$. On the other hand, for $y = 0$ and $x \geq X$, one has

$$\begin{aligned} w_y &= (u_{xy} - \gamma u_x) - ((\Gamma_{\alpha_1})_y - \gamma\Gamma_{\alpha_1}) + C((\Gamma_{\tilde{\alpha}})_y - \gamma\Gamma_{\tilde{\alpha}}) \\ &= (u_y - \gamma u)_x - (\delta_1 - \gamma)\Gamma_{\alpha_1} \\ &= -2uu_x - (\delta_1 - \gamma)\Gamma_{\alpha_1} \leq 0. \end{aligned}$$

This contradiction shows that w cannot have any negative values in Ω, and so the claim has been proved.

It now follows immediately from the claim that there exists $D_{\tilde{\alpha}} > 0$ such that $u_x - \Gamma_{\alpha_1} \geq -D_{\tilde{\alpha}}\Gamma_{\tilde{\alpha}}$ for all $(x,y) \in [0,\infty) \times [0,1]$. Since Γ_{α_1} is a positive function, this yields the desired estimate on $[0,\infty) \times [0,1]$. The corresponding estimate on $(-\infty,0] \times [0,1]$ is proved similarly. □

Now to continue with the proof of the Theorem, let $v(x,y)$ denote the harmonic conjugate of u on S given by the formula

$$v(x,y) = -\int_0^x u_y(s,0)\ ds + \int_0^y u_x(x,t)\ dt.$$

By Lemmas 1 and 2(vi), $u_y(s,0) = -N\phi(s)$ is an L^1 function of s, and $u_x(x,t)$ vanishes, uniformly in t, as $|x| \to \infty$. Therefore v is a bounded function on S. Define the function $\mu(x,y)$ on $\bar{S}$ by $\mu = u_x + uv$. Then μ is harmonic on S, and from Lemma 4, Lemma 7, and the boundedness of v, it follows that there exists a constant $r > 0$ such that $\mu \geq -ru$ on S.

Let $R = \inf\{r \in \mathbb{R} : \mu \geq -ru \text{ on } S\}$, and let ρ be the function defined on S by $\rho = \mu + Ru = u_x + u(v+R)$. Note that $\rho \geq 0$ on $\bar{S}$.

Lemma 8. *The function ρ is identically zero on $\bar{S}$.*

Proof. For each natural number n define $\rho_n = \mu + (R - \frac{1}{n})u$. From the definition of R it follows that ρ_n attains a negative value at some point in S. Fix δ (independent of n) so that $-1 < \delta < \gamma$, and define $w_n = \dfrac{\rho_n}{1+\delta y}$ on $\bar{S}$. Then w_n satisfies a maximum principle on S and vanishes for $y = 1$ and for $|x| \to \infty$. Hence, since w_n takes negative values in S, there must be some point $(x_n, 0)$ in $\bar{S}$ at which w_n attains its negative minimum and at which $(w_n)_y > 0$. Now for $y = 0$ it follows from the Cauchy-Riemann equations that

$$\begin{aligned}
(w_n)_y &= (\rho_n)_y - \delta\rho_n \\
&= [(R - \frac{1}{n})u + u_x + uv]_y - \delta\rho_n \\
&= (\gamma - u)[(R - \frac{1}{n})u + u_x + uv] - \delta\rho_n \\
&= (\gamma - \delta - u)\rho_n,
\end{aligned}$$

where use has been made of the relation $u_y = \gamma u - u^2$. Since $(w_n)_y > 0$ and $\rho_n < 0$ at $(x_n, 0)$, it follows that $u(x_n, 0) > \gamma - \delta$. This proves that the sequence $\{u(x_n, 0)\}$ is bounded away from 0. Hence, since $u(x, 0) \to 0$ as $|x| \to \infty$, the sequence $\{x_n\}$ must be bounded in $\mathbb{R}$, and so has a subsequence converging to a limit $x^* \in \mathbb{R}$. Now taking the limit of the inequality $\rho_n(x_n, 0) < 0$ as $n \to \infty$ yields that $\rho(x^*, 0) \leq 0$. Since $\rho \geq 0$ by the choice of R, $\rho(x^*, 0) = 0$.

Since $\rho \geq 0$ on $\bar{S}$, the function ρ takes its minimum value on $\bar{S}$ at the point $(x^*, 0)$. Therefore, if ρ is not identically zero on $\bar{S}$ then the Hopf boundary point lemma for harmonic functions implies that $\rho_y(x^*, 0) > 0$. But this is false, as $\rho_y = [Ru + u_x + uv]_y = (\gamma - u)[Ru + u_x + uv] = (\gamma - u)\rho$ implies that $\rho_y(x^*, 0) = 0$. Thus the Lemma has been proved. □

The proof of the main Theorem may now be completed. Define a function f of the complex variable $z = x + iy$ by setting $f(z) = u(x, y) + i[v(x, y) + R]$. Since u and $v + R$ are harmonic conjugates on S, then f is holomorphic on S. Moreover, the function $G(z) = \frac{\partial f}{\partial z} - \frac{i}{2} f^2$, which is also holomorphic on S, has real part $u_x + u(v + R) = \rho = 0$ on S, and hence must be identically equal on S to a purely imaginary constant $2iM$ (where $M \in \mathbb{R}$). Therefore f is a solution of the first-order complex differential equation

$$\frac{\partial f}{\partial z} = \frac{i}{2}[f^2 + M]$$

which has already arisen, in a similar context, in the study of Benjamin-Ono solitary waves appearing in [2] and [3].

The equation for f is easily integrated, to yield the solutions

$$f(z) = \begin{cases} \sigma \tan(\frac{i\sigma}{2}(z + C) & \text{(if } M = \sigma^2 > 0) \\ -\sigma \tanh(\frac{i\sigma}{2}(z + C) & \text{(if } M = -\sigma^2 < 0) \\ \frac{-2}{i(z+C)} & \text{(if } M = 0) \end{cases}$$

where C is an arbitrary complex constant of integration. The solitary wave profile $\phi(x)$ may be recovered from f by taking $\phi(x) = u(x, y)|_{y=0} = \text{Real } f(x + iy)|_{y=0}$. Examination of the various possibilities shows that in order for ϕ to be a function in $L^2(\mathbb{R})$, M cannot be negative. In the case $M = 0$, one finds after letting $C = p + iq$

that $u(x,y) = \dfrac{2(y+q)}{(x+p)^2+(y+q)^2}$ and $\phi(x) = \dfrac{2q}{(x+p)^2+(y+q)^2}$ (the Benjamin-Ono solitary wave, cf. [2] and [3]). In order for u to satisfy the requirement that u vanish for $y=1$, one must take $q=-1$, in which case the function u is not C^2 on $\bar{S}$. Therefore the only remaining possibility is that M be positive, and u is then given by

$$u(x,y) = \frac{-\sigma \sin(\sigma(y+q))}{\cosh(\sigma(x+p)) + \cos(\sigma(y+q))}.$$

Here the condition that u vanish for $y=1$ is met only if $q=-1$, in which case

$$\phi(x) = u(x,0) = \frac{\sigma \sin \sigma}{\cosh \sigma(x+p) + \cos \sigma}.$$

The value of σ is determined by the condition that $u_y = \gamma u - u^2$; and an easy computation shows that this implies $\sigma \cot \sigma = -\gamma$. The rest of the proof is identical to the treatment of the Benjamin-Ono equation appearing on pp. 112–113 of [2]. This completes the proof of the Theorem.

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