Positivity Properties and Uniqueness of Solitary Wave Solutions of the Intermediate Long-Wave Equation

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1. INTRODUCTION

The intermediate long wave (ILW) equation was first proposed in [9] and [12] as a model equation for long internal gravity waves at the interface between two fluids of different densities, each of finite depth H. Using the rescaled variables introduced in [12], the ILW equation can be written in the form

$$u_t + 2uu_x - (N_H u)_x + \frac{1}{H} u_x = 0, \tag{1.1}$$

where the "dispersion operator" N_H is the Fourier multiplier operator defined by

$$(N_H u)\hat{\ }(k) = (k \coth kH)\hat{u}(k).$$

(Here and throughout the paper, circumflexes are used to denote Fourier transforms in the x variable: thus $\hat{u}(k,t)$ denotes $\int_{-\infty}^{\infty} e^{ikx} u(x,t)\, dx$.)

An explicit family of solitary wave solutions of (1.1) was given by Joseph in [9]. These have the form $u(x,t) = \phi_{C,H}(x - Ct)$, where

$$\phi_{C,H}(z) = \left[\frac{a \sin aH}{\cosh az + \cos aH}\right].$$

Here $C > 0$ is arbitrary, and $a \in (0, \pi/H)$ is determined uniquely by the equation $aH \cot aH = (1 - CH)$. It was subsequently found that these solutions have solitonic properties like those of Korteweg-de Vries solitary waves ([10], [11],[13]). This was perhaps to be expected, in view of the role that ILW plays as an "interpolator" between two well-known soliton equations: the Korteweg-de Vries equation and the Benjamin-Ono equation, which are obtainable from ILW in the limits $H \to 0$ and $H \to \infty$ respectively ([1],[5],[14]).

Recently, it has been proved ([6]) that, for a given choice of $C > 0$ and $H > 0$, Joseph's solitary wave $\phi_{C,H}$ is, up to translations, the only L^2 travelling-wave solution of (1.1). The proof given in [6], which is based on Amick and Toland's proof of uniqueness of L^2 and periodic travelling wave solutions of the Benjamin-Ono equation ([7],[8]), treats the ILW operator as a Dirichlet-to-Neumann map for a strip in the complex plane, and makes crucial use of maximum-principle arguments. It is the purpose of this note to provide an alternate proof of uniqueness of ILW (and Benjamin-Ono) solitary waves which makes no use of complex analysis. Instead, it will be seen here that uniqueness of solitary waves can be deduced directly from two simple properties of the dispersion operator N_H: namely, the positivity of the resolvent $(N_H + \gamma)^{-1}$, and the well-known product identity stated below in Lemma 3.

Although Amick and Toland's original proof of uniqueness of Benjamin-Ono solitary waves is remarkable for its seemingly inexorable progression from one geometric idea to another, it is not easy to understand exactly which properties of the Benjamin-Ono equation are responsible for the success of the method. It is hoped that the alternate approach presented here will shed some light on this question. This approach also provides yet another illustration of the importance of positivity properties of the dispersion operator for the study of solitary waves. (See [3], [4], and [15] for applications of positive-operator theory to the questions of existence and stability of solitary waves.)

2. STATEMENT OF THE UNIQUENESS THEOREM

Attention in this paper will be confined to solutions of (1.1) which vanish in some sense as $|x| \to \infty$; thus excluding, for example, the periodic solutions discussed in [2]. In view of the results of [1] on the well-posedness of the initial-value problem for (1.1), the L^2 based Sobolev spaces H^s form a natural setting for the present study. (In this paper, H^s will denote the Hilbert space of all tempered distributions f whose Fourier transforms $\hat{f}$ are functions satisfying $\left(\int_{-\infty}^{\infty} |\hat{f}(k)|^2 (1 + |k|^2)^s \, dk\right)^{1/2} = \|f\|_s < \infty$. Use will also be made of the spaces $L^p = \{f : \left(\int_{-\infty}^{\infty} |f|^p\right)^{1/p} < \infty\}$ for $1 \leq p < \infty$ and $L^\infty = \{f : \text{ess sup } |f(x)| < \infty\}$; and the Schwartz class $\mathbf{S}$ of functions f such that

$\lim_{|x|\to\infty} |x|^m \left(\frac{d^n f}{dx^n}\right)(x) = 0$ for all nonnegative integers m and n.)

By a "travelling wave" solution of (1.1) is understood any solution of the form $u(x,t) = \phi(x - Ct)$, where $C \in \mathbf{R}$. Usually a nontrivial (i. e., not identically zero) travelling wave solution which vanishes at $\pm\infty$ is referred to as a solitary wave. If ϕ is sufficiently regular and vanishes at $\pm\infty$ (say, $\phi \in H^1$), then substituting $u = \phi(x - Ct)$ into (1.1) and integrating once yields the equation

$$(N_H + \gamma)\phi = \phi^2 \tag{2.1}$$

where $\gamma = C - (1/H)$. This will be considered the defining equation for solitary waves. Since both sides of (2.1) are well-defined as tempered distributions for any $\phi \in L^2$ (note $N_H : H^s \to H^{s-1}$ for all $s \in \mathbf{R}$), one can extend the definition of solitary wave to include any nontrivial L^2 solution of (2.1).

The following uniqueness result for (2.1) is proved in [6].

THEOREM. Let $H > 0$ and $C > 0$ be given, and let $\gamma = C - (1/H)$. If $\phi \in L^2$ is a nontrivial solution of (2.1), then there exists $b \in \mathbf{R}$ such that $\phi(x) \equiv \phi_{C,H}(x + b)$, where $\phi_{C,H}$ is as defined in Section 1.

The goal of the present paper is to provide an alternate proof, which shows that the theorem is in fact an immediate consequence of the two special properties of the operator N_H stated in Lemmas 1 and 3 of the next section.

3. TWO PROPERTIES OF THE DISPERSION OPERATOR

For ease of notation, the subscript H will henceforth be dropped from N_H. Also, the notation $m(k)$ will occasionally be used for the multiplier function $k \coth kH$.

LEMMA 1. Let $\gamma > -(1/H)$ be given. Then the operator $(N + \gamma) : H^{s+1} \to H^s$ is invertible with bounded inverse for every $s \geq 0$. Moreover, for all $f \in H^s$, one has $\left[(N + \gamma)^{-1} f\right](x) = \int_{-\infty}^{\infty} K(x - y) f(y)\, dy$, where the kernel $K(x)$ belongs to L^p for every $p \in [1, \infty)$, and satisfies $K(x) > 0$ for all $x \in \mathbf{R}$.

PROOF: For every $f \in H^s$, the equation $(N+\gamma)g = f$ has the unique solution $g \in H^{s+1}$ given by $\hat{g}(k) = (m(k) + \gamma)^{-1} \hat{f}(k)$; and clearly $\|g\|_{s+1} \leq C\|f\|_s$. Therefore the kernel K of $(N + \gamma)^{-1}$ is given by

$$K(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\frac{1}{k \coth kH + \gamma}\right] e^{-ikx}\, dk.$$

A residue calculation using Jordan's Lemma shows that

$$K(x) = \sum_{j=0}^{\infty} e^{-\theta_j |x|/H} \left[\frac{2\sin^2\theta_j}{2\theta_j - \sin(2\theta_j)} \right]$$

where $\{\theta_j\}_{j=0,1,2,\ldots}$ are the positive solutions of $\theta_j \cot\theta_j + \gamma H = 0$. It follows immediately that $K(x) > 0$ for $x \in \mathbf{R}$. Also, since $j\pi < \theta_j < (j+1)\pi$ for all $j \geq 0$, it follows that for some constants A, A_1, A_2 (independent of x) one has

$$\begin{aligned} |K(x)| &\leq A \left[e^{-\theta_0|x|/H} + \sum_{j=1}^{\infty} \left(\frac{1}{j}\right) e^{-j\pi|x|/H} \right] \\ &= A \left[e^{-\theta_0|x|/H} + |\log\left(1 - e^{-\pi|x|/H}\right)| \right] \\ &\leq \begin{cases} A_1 |\log|x|| & \text{for } |x| \leq 1 \\ A_2\, e^{-\theta_0|x|/H} & \text{for } |x| \geq 1 \end{cases} \end{aligned}$$

Therefore $K \in L^p$ for every $1 \leq p < \infty$. This completes the proof of the lemma.

Next we prove the product identity (3.1) for N. For Schwarz-class functions f and g, this identity follows immediately from a Fourier transform calculation. (It may also be derived as a consequence of the fact that N represents a Dirichlet-to-Neumann map for a horizontal strip in the complex plane, see e.g. [13].) However, here it will be required to establish the identity for functions f and g which do not necessarily vanish rapidly at infinity. For this purpose, the following lemma will be useful.

LEMMA 2. Suppose $f \in L^2$, $g \in H^2$, and ψ is in the Schwarz class $\mathbf{S}$. Then

$$\int_{-\infty}^{\infty} f(x) \left(\int_0^x Ng(y)\, dy \right) \hat{\psi}(x)\, dx =$$

$$= \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \coth kH\; \hat{g}(k) \left\{ \int_{-\infty}^{\infty} \left[\hat{f}(y-k) - \hat{f}(y) \right] \psi(y)\, dy \right\}\, dk.$$

PROOF: The left-hand side of the equation in the lemma may be rewritten as

$$\int_{-\infty}^{\infty} f(x) \left\{ \int_0^x \frac{1}{2\pi} \int_{-\infty}^{\infty} m(k)\hat{g}(k) e^{-iky}\, dk\, dy \right\} \hat{\psi}(x)\, dx.$$

Since $g \in H^2$, then $k\ \hat{g}(k) \in L^1(dk)$, and so Fubini's theorem may be applied to the integral in braces, yielding

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left\{ \int_{-\infty}^{\infty} m(k)\hat{g}(k) \left[\frac{e^{-ikx} - 1}{-ik} \right] dk \right\} f(x)\hat{\psi}(x)\, dx.$$

Now, since

$$\left| f(x)\hat{\psi}(x) \left[\frac{e^{-ikx}-1}{-ik} \right] \right| = \left| x f(x)\hat{\psi}(x) \left[\frac{e^{-ikx}-1}{-ikx} \right] \right| \le C |x f(x)\hat{\psi}(x)| \in L^1(dx),$$

another application of Fubini's theorem yields

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} m(k)\hat{g}(k) \left\{ \int_{-\infty}^{\infty} f(x)\hat{\psi}(x) \left[\frac{e^{-ikx}-1}{-ik} \right] \, dx \right\} \, dk.$$

Finally, upon applying Parseval's identity to the inner integral in braces, one obtains the right-hand side of the equation in the statement of the lemma, thus completing the proof.

LEMMA 3. If $f, g \in H^2$ and $Nf, Ng \in L^1 \cap L^2$, then

$$f \cdot g' + f' \cdot g + N \left[f \left(\int_0^x Ng \right) + g \left(\int_0^x Nf \right) \right] - Nf \left(\int_0^x Ng \right) - Ng \left(\int_0^x Nf \right) = 0. \quad (3.1)$$

PROOF: Let $h(x) = f(x) \left(\int_0^x Ng(y) \, dy \right)$. From the assumptions on f and g it follows that $Nh \in L^2$. Therefore, for every $\psi \in \mathbf{S}$, one has

$$\int_{-\infty}^{\infty} Nh(x)\hat{\psi}(x) \, dx = \int_{-\infty}^{\infty} h(x)\hat{\theta}(x) \, dx,$$

where $\theta \in \mathbf{S}$ is defined by $\theta(x) = m(x)\psi(x)$. So from Lemma 2 it follows that

$$\int_{-\infty}^{\infty} N \left(f \cdot \left(\int_0^x Ng \right) \right) (x) \; \hat{\psi}(x) \, dx = \int_{-\infty}^{\infty} f(x) \left(\int_0^x Ng(y) \, dy \right) \hat{\theta}(x) \, dx =$$

$$= \frac{-1}{2\pi i} \int_{-\infty}^{\infty} m(k)\hat{g}(k) \left\{ \int_{-\infty}^{\infty} \left[\hat{f}(y-k) - \hat{f}(y) \right] m(y)\psi(y) \, dy \right\} \, dk.$$

Also, by Lemma 2,

$$\int_{-\infty}^{\infty} Nf(x) \left(\int_0^x Ng(y) \, dy \right) \hat{\psi}(x) \, dx =$$

$$= \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \coth kH \; \hat{g}(k) \left\{ \int_{-\infty}^{\infty} \left[m(y-k)\hat{f}(y-k) - m(y)\hat{f}(y) \right] \psi(y) \, dy \right\} \, dk.$$

Hence

$$\int_{-\infty}^{\infty} \left[N \left(f \cdot \left(\int_0^x Ng \right) \right) - Nf \left(\int_0^x Ng \right) \right] \hat{\psi}(x) \, dx =$$

$$= \frac{-1}{2\pi i} \int_{-\infty}^{\infty} \psi(y) \int_{-\infty}^{\infty} \hat{g}(k)\hat{f}(y-k) \coth kH \left\{ m(y) - m(y-k) \right\} \, dk \, dy,$$

where the change in the order of integration in the last expression is justified by the absolute integrability of the integrand in $\mathbf{R}^2$.

Since $\psi \in \mathbf{S}$ was arbitrary, the preceding computation shows that, for all $y \in \mathbf{R}$,

$$\left[N\left(f \cdot \left(\int_0^x g\right)\right) - Nf\left(\int_0^x Ng\right)\right]^{\widehat{}}(y) =$$

$$= \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \hat{g}(k)\hat{f}(y-k)\coth kH\left\{m(y) - m(y-k)\right\}\ dk.$$

Now switching the roles of f and g and making the change of variables from k to $\tilde{k} = y - k$ yields the result

$$\left[N\left(g \cdot \left(\int_0^x f\right)\right) - Ng\left(\int_0^x Nf\right)\right]^{\widehat{}}(y) =$$

$$= \frac{-1}{2\pi i}\int_{-\infty}^{\infty} \hat{g}(k)\hat{f}(y-k)\coth(y-k)H\left\{m(y) - m(k)\right\}\ dk.$$

Adding the two previous equations to the equation

$$[fg' + f'g]^{\widehat{}}(y) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \hat{g}(k)\hat{f}(y-k)y\ dk,$$

one finally obtains that the Fourier transform of the left-hand side of (3.1) is given as a function of y by

$$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \hat{g}(k)\hat{f}(y-k)\ Q(y,k)\ dk,$$

where

$$Q(y,k) = y - \coth kH\ (m(y) - m(y-k)) - \coth((y-k)H)\ (m(y) - m(k)).$$

But an elementary computation shows that $Q(y,k) = 0$ for all y and k. Thus the proof of the lemma is complete.

Remark: The existence of some identity such as (3.1) seems to be required in order for an equation of the form (1.1) to have a "completely integrable" structure; see [1], [2], [11], [13].

4. A PROOF OF THE UNIQUENESS THEOREM

In this section, the uniqueness theorem stated in section 2 is derived as a consequence of Lemmas 1 and 3. The plan of the proof is as follows: first, in Lemma 4, regularity and positivity properties are derived for solutions of (2.1). Next, the identity (3.1) and the Krein-Rutman theorem for positive operators is used to reduce (2.1) to the ordinary differential equation (4.1). Standard uniqueness theory for ordinary differential equations then implies the desired result.

LEMMA 4. Let $\phi \in L^2$ be a nontrivial solution of (2.1). Then ϕ and $N\phi$ and all their derivatives are in L^p for $1 \le p \le \infty$. Also, $\phi(x) > 0$ for all $x \in \mathbf{R}$.

PROOF: First it will be proved by induction on k that $(\frac{d}{dx})^k(\phi) \in L^p$ for $1 \le p < \infty$. From (2.1) and Lemma 1, one has $\phi = K * \phi^2$, where $K \in L^p$ for $1 \le p < \infty$. Since $\phi \in L^2$, then $\phi^2 \in L^1$, and so it follows from Young's convolution inequality that $\phi \in L^p$ for $1 \le p < \infty$. Now assume that all the derivatives of ϕ up to order k are in L^p for $1 \le p < \infty$. Then from the Leibniz differentiation formula and the Cauchy-Schwarz inequality, it follows that $(\frac{d}{dx})^k(\phi^2) \in L^2$. Hence, by (2.1) and Lemma 1, $(\frac{d}{dx})^{k+1}(\phi) = \frac{d}{dx}(N+\gamma)^{-1}(\frac{d}{dx})^k(\phi^2) \in L^2$. Another use of Leibniz' formula now yields that $(\frac{d}{dx})^{k+1}(\phi^2) \in L^1$. Since $(\frac{d}{dx})^{k+1}(\phi) = K * (\frac{d}{dx})^{k+1}(\phi^2)$, it follows from Young's inequality that $(\frac{d}{dx})^{k+1}(\phi) \in L^p$ for $1 \le p < \infty$. So the inductive argument is complete.

It follows from what has just been proved that for all k, $(\frac{d}{dx})^k(\phi)$ is in $H^1(\mathbf{R})$ and hence in $L^\infty(\mathbf{R})$ as well. Also, the assertions of the Lemma concerning $N\phi$ follow from the fact that $N\phi = \phi^2 - \gamma\phi$.

Finally, to prove that if ϕ is not identically zero then $\phi(x) > 0$ for all $x \in \mathbf{R}$, one merely notes that, by Lemma 1, $\phi(x) = \int_{-\infty}^{\infty} K(x-y)\phi^2(y)\, dy$, where the kernel K is a strictly positive function on $\mathbf{R}$. This completes the proof of the Lemma.

To continue now with the proof of the main result, let ϕ be an arbitrary nontrivial L^2 solution of (2.1). By Lemma 4, $\phi > 0$ on $\mathbf{R}$, and so a Hilbert space Y may be defined to consist of all real-valued measurable functions g on $\mathbf{R}$ such that $\int_{-\infty}^{\infty} |g(x)|^2\phi(x)\, dx < \infty$, with norm furnished by the inner product $\langle g, h\rangle_Y = \int_{-\infty}^{\infty} g(x)h(x)\phi(x)\, dx$. The norm of g in Y is denoted by $\|g\|_Y$.

Next, define a linear operator $T : Y \to Y$ by

$$Tg = (N+\gamma)^{-1}(\phi g).$$

The spectral properties of T on Y have already been studied in some detail in [3] and [4]. Here we require only the following fact, which is a consequence of Lemmas 1 and 4 and the classical Krein-Rutman theorem on positive operators: T is a self-adjoint compact operator on Y, and has a positive simple eigenvalue λ_0 with a corresponding strictly positive eigenfunction $f_0(x)$. (For a proof, see Proposition 2.5.b of [3] and Lemma 8.a. of [4].)

On the other hand, the strictly positive function ϕ is also an eigenfunction of T, for the eigenvalue 1, as can be seen by rewriting (2.1) in the form

$$\phi = (N+\gamma)^{-1}(\phi^2) = T\phi.$$

Therefore λ_0 must equal 1; for otherwise, it would follow from the orthogonality of eigenspaces of self-adjoint operators that $\langle \phi, f_0\rangle_Y = 0$, which is impossible since ϕ and f_0 are both strictly positive on $\mathbf{R}$.

Now consider the function $\psi \in Y$ defined by $\psi(x) = \phi'(x) + \phi(x) \cdot \int_0^x N\phi(y)\, dy$. We claim that ψ is also an eigenfunction of T for the eigenvalue $\lambda_0 = 1$. To see this, write

$$\begin{aligned}(N+\gamma)\psi &= (N+\gamma)\phi' + (N+\gamma)(\phi \int_0^x N\phi) \\ &= \frac{d}{dx}\left[\phi^2\right] + N(\phi \int_0^x N\phi) + \gamma\phi \int_0^x N\phi \\ &= \phi\phi' + (N\phi)\int_0^x N\phi + \gamma\phi\int_0^x N\phi \\ &= \phi\phi' + \phi^2 \int_0^x N\phi = \phi\psi,\end{aligned}$$

where use has been made of (2.1) and Lemma 3 (with $f = g = \phi$). Hence

$$\psi = (N+\gamma)^{-1}\phi\psi = T\psi,$$

as claimed.

From the simplicity of the eigenvalue $\lambda_0 = 1$ of T, it now follows that ψ is a scalar multiple of ϕ: that is, there exists $\beta \in \mathbf{R}$ such that $\psi = \beta\phi$. Hence $\phi' + \phi\int_0^x N\phi = \beta\phi$. Solving for $N\phi$, one obtains that $N\phi = (-\phi'/\phi)'$. Substituting in (1.2) then yields the ordinary differential equation

$$\gamma\phi - \phi^2 = \left(\frac{\phi'}{\phi}\right)'. \tag{4.1}$$

To analyze equation (4.1), multiply by ϕ'/ϕ and integrate to obtain

$$(\phi')^2 = \phi^2(D + 2\gamma\phi - \phi^2), \tag{4.2}$$

where D is the constant of integration. Now since $\phi \in H^1$ by Lemma 4, then $\hat{\phi} \in L^1$ and, by the Riemann-Lebesgue lemma, $\phi \to 0$ as $|x| \to \infty$. It follows then, from taking the limit of (4.2) as $|x| \to \infty$, that $D \geq 0$.

Consider first the case $D > 0$. Then by (4.2),

$$\phi' = \pm\phi\sqrt{(P_+ - \phi)(P_- - \phi)}, \tag{4.3}$$

where $P_\pm = \gamma \pm \sqrt{D + \gamma^2}$, $P_+ > 0$, and $P_- < 0$. Let $\phi(0) = r$. From (4.3) it is clear that $r \in (0, P_+]$. If $r < P_+$, then an explicit integration of (4.3), together with the basic uniqueness theorem for ordinary differential equations, implies that, at least for x near 0,

$$\phi(x) = \left[\frac{a \sin aH}{\cosh a(x \pm b) + \cos aH}\right], \tag{4.4$_\pm$}$$

where $a = \sqrt{D}$, $H \in (0, \pi/a)$ is chosen so that $a \cot aH = -\gamma$, and $b > 0$ is uniquely determined by $\phi(0) = r$, with the positive or negative sign being used in $(4.4)_\pm$ according as to whether $\phi'(0) < 0$ or $\phi'(0) > 0$ respectively.

If $\phi'(0) > 0$, then the solution $(4.4)_-$ is valid at least from $x = -\infty$ to the point $x = b$, where $\phi = P_+$ and the solution of (4.3) ceases to be unique. However, at this point (4.3) gives $\phi' = 0$, and so (4.1) gives $\phi'' = \gamma - P_+ < 0$; showing that ϕ has a strict local maximum at $x = b$. Therefore $\phi(x) < P_+$ for $x > b$, and again uniqueness of solutions of (4.3) shows that ϕ is given by $(4.4)_-$ for all $x \in (b, \infty)$ as well. The same argument shows that $(4.4)_+$ holds for all $x \in \mathbf{R}$ if $\phi'(0) < 0$. Finally, if $\phi(0) = r = P_+$ and $\phi'(0) = 0$, then again (4.1) implies $\phi''(0) < 0$, and (4.4) is seen to hold for all $x \in \mathbf{R}$ with $b = 0$.

It remains to consider the case $D = 0$. But here the same argument as in the preceding paragraph shows that any positive C^2 solution of (4.1) and (4.2) must have the form

$$\phi(x) = \frac{2\gamma}{1 + \gamma^2(x + b)^2} \tag{4.5}$$

for some $b \in \mathbf{R}$.

It has now been shown, therefore, that every solution of (4.1) must be given by either (4.4) or (4.5), where the constants $a > 0$ and $H > 0$ in (4.4) are related by $a \cot aH = -\gamma$. It is easily verified that the function in (4.5) does not satisfy (2.1) (but see the remark below). Therefore, for given $H > 0$ and $\gamma = C - (1/H)$, the only solutions of (2.1) are given by (4.4) with b arbitrary and a determined by $a \cot aH = C - (1/H)$. Hence $\phi(x) = \phi_{C,H}(x \pm b)$, and the proof of the theorem is complete.

Remark: A similar proof can be used to recover Amick and Toland's result in [7] on the uniqueness of Benjamin-Ono solitary waves. The Benjamin-Ono equation is

$$u_t + 2uu_x - (N_\infty u)_x = 0$$

where $(N_\infty u)\hat{}(k) = |k|\hat{u}(k)$. From the identity

$$\text{sign } k \; [1 - \text{sign } y \text{ sign } (y - k)] = [\text{sign } y - \text{sign } (y - k)]$$

(valid for all real y and k), one easily derives the well-known product formula

$$fg + H(f \cdot Hg + g \cdot Hf) - Hf \cdot Hg = 0 \tag{4.6}$$

(valid for all f and g in L^2), where H denotes the Hilbert transform: $(Hf)\hat{}\,(k) = (i \text{ sign } k)\hat{f}(k)$. Since $\frac{d}{dx}H = N_\infty$, differentiation of (4.6) yields (3.1) with N replaced by N_∞. It may be further seen that all the assertions of Lemma 1 hold with $N + \gamma$ replaced by $N_\infty + \gamma$, where $\gamma > 0$ is arbitrary. Hence the same argument as above leads to the conclusion that every L^2 solution of the Benjamin-Ono solitary-wave equation $(N_\infty + \gamma)\phi = \phi^2$ is also a solution of (4.1) and (4.2). From the above analysis of (4.2) it then follows that ϕ must in fact be given by (4.5).

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