Existence and stability of ground-state solutions of a Schrödinger-KdV system

John Albert
Department of Mathematics
University of Oklahoma
Norman, OK 73019

Jaime Angulo Pava
Department of Mathematics
IMECC-UNICAMP
C.P. 6065. CEP 13083-970
Campinas São Paulo, Brazil

We consider the coupled Schrödinger-KdV system

$$\begin{cases} i(u_t + c_1 u_x) + \delta_1 u_{xx} = \alpha uv \\ v_t + c_2 v_x + \delta_2 v_{xxx} + \gamma(v^2)_x = \beta(|u|^2)_x, \end{cases}$$

which arises in various physical contexts as a model for the interaction of long and short nonlinear waves. Ground states of the system are, by definition, minimizers of the energy functional subject to constraints on conserved functionals associated with symmetries of the system. In particular, ground states have a simple time dependence because they propagate via those symmetries. For a range of values of the parameters $\alpha, \beta, \gamma, \delta_i, c_i$, we prove the existence and stability of a two-parameter family of ground states associated with a two-parameter family of symmetries.

1. Introduction

In this paper we prove existence and stability results for ground-state solutions to the system of equations

$$\begin{cases} i(u_t + c_1 u_x) + \delta_1 u_{xx} = \alpha uv \\ v_t + c_2 v_x + \delta_2 v_{xxx} + \gamma(v^2)_x = \beta(|u|^2)_x, \end{cases} \tag{1.1}$$

where u is a complex-valued function of the real variables x and t, v is a real-valued function of x and t, and the constants $c_i, \delta_i, \alpha, \beta, \gamma$ are real. We consider here only the pure initial-value problem for (1.1), in which initial data $(u(x,0), v(x,0)) = (u_0(x), v_0(x))$ is posed for $-\infty < x < \infty$, and a solution $(u(x,t), v(x,t)$ is sought for $-\infty < x < \infty$ and $t \geq 0$. Well-posedness results for the pure initial-value problem for (1.1) and certain of its variants have appeared in [7,21,34]; we cite below in Section 5 the specific results we will need here.

Systems of the form (1.1) appear as models for interactions between long and short waves in a variety of physical settings. For example, Kawahara et al. [23] derived (1.1) as a model for the interaction between long gravity waves and capillary waves on the surface of shallow water, in the case when the group velocity of the capillary wave coincides with the velocity of the long wave. In [30,32], a system of equations is derived for resonant ion-sound/Langmuir wave interactions in plasmas which reduces to (1.1) under the assumption that the ion-sound wave is unidirectional. Similarly, one can obtain (1.1) as the unidirectional reduction of a model for the resonant interaction of acoustic and optical modes in a diatomic lattice [38].

In the applications mentioned in the preceding paragraph, all the constants appearing in (1.1) are typically non-zero. On the other hand, (1.1) with $\delta_2 = \gamma = 0$ was derived in [16] and [19] as a model for the interaction between long and short water waves, and appears as well in the plasma physics literature (see, for example, [22, 37]). The presence or absence of the terms containing δ_2 and γ is determined by the scaling assumptions made in the derivation of the equations. For a discussion of the role of the scaling assumptions in the derivation of equations such as (1.1), the reader may consult [10] or [17].

If $\delta_2 \neq 0$ in (1.1), then by making appropriate use of the transformations $x \to \theta x$, $t \to \theta t$, $x \to x+t$, $u \to \theta u$, $u \to \overline{u}$, and $u \to e^{i(\theta x - \theta^2 t)}u$, where $\theta \in \mathbb{R}$, we can reduce (1.1) to either

$$\begin{cases} iu_t + u_{xx} = -uv \\ v_t + 2v_{xxx} + 3q(v^2)_x = -(|u|^2)_x \end{cases} \tag{1.2}$$

or

$$\begin{cases} iu_t + u_{xx} = -uv \\ v_t - 2v_{xxx} + 3q(v^2)_x = -(|u|^2)_x, \end{cases} \tag{1.3}$$

where $q \in \mathbb{R}$. System (1.3) is the form that arises in [5,30,32]; its analysis is complicated by the fact that the associated energy functional, analogous to the energy $E(u,v)$ defined below, is not positive definite. In this paper we do not consider (1.3), and we further assume that $q > 0$ in (1.2). The case $q > 0$ in (1.2) arises, for example, when modelling interactions between internal and surface gravity waves in a two-layer fluid, provided the ratio of the depth of the upper layer to the depth of the lower layer is less than a certain critical value [17].

We will also have occasion below to consider the case when $\delta_2 = \gamma = 0$ in (1.1). In this case (1.1) can be reduced to the form

$$\begin{cases} iu_t + u_{xx} = -uv \\ v_t = -(|u|^2)_x. \end{cases} \tag{1.4}$$

System (1.4) is of independent mathematical interest because it has been found to have a completely integrable structure. In particular, it has an inverse scattering transform and explicit N-soliton solutions [28, 29, 37]. (By contrast, equations (1.2) and (1.3) do not have N-soliton solutions [9]).

The system (1.2) can be written in Hamiltonian form as

$$(u_t, v_t) = J\ \delta E(u,v), \tag{1.5}$$

where J is the antisymmetric operator defined by $J(w,z) = ((-i/2)w, z_x)$, and $E(u,v)$, the Hamiltonian functional, is defined by

$$E(u,v) = \int_{-\infty}^{\infty} \left(|u_x|^2 + v_x^2 - v|u|^2 - qv^3\right)\, dx.$$

The notation δE in (1.5) refers to the Fréchet derivative, or generalized gradient, of E. Since the Hamiltonian E is invariant under time translations, it is a conserved functional for the flow defined by (1.2): i.e., when applied to sufficiently regular solutions $u(x,t)$, $v(x,t)$ of (1.2), E is independent of t. There are also two other conserved functionals of (1.2) associated with symmetries: namely,

$$G(u,v) = \int_{-\infty}^{\infty} v^2\, dx - 2\ \mathrm{Im} \int_{-\infty}^{\infty} u\overline{u}_x\ dx,$$

which arises from the invariance of (1.2) under space translations $x \to x + \theta$, and

$$H(u) = \int_{-\infty}^{\infty} |u|^2\ dx,$$

which arises from the invariance of (1.2) under phase shifts $u \to e^{i\theta}u$.

Equations (1.4) can also be rewritten in Hamiltonian form as

$$(u_t, v_t) = J\ \delta K(u,v), \tag{1.6}$$

where J is as above and K is defined by

$$K(u,v) = \int_{-\infty}^{\infty} \left(|u_x|^2 - v|u|^2\right)\ dx.$$

The functionals $G(u,v)$ and $H(u)$ defined above are conserved functionals for (1.4) as well.

Bound-state solutions of (1.2) or (1.4) are, by definition, solutions of the form

$$u(x,t) = e^{i\omega t}h(x - ct), \qquad v(x,t) = g(x - ct), \tag{1.7}$$

where h and g are functions which vanish at infinity in some sense (usually h and g are in $H^1(\mathbb{R})$), and ω and c are real constants. It is easy to see that $u(x,t)$ and $v(x,t)$ as defined in (1.7) are solutions of (1.2) if and only if (h,g) is a critical point for the functional $E(u,v)$, when $u(x)$ and $v(x)$ are varied subject to the constraints that $G(u,v)$ and $H(u)$ be held constant (see Section 5 below). If (h,g) is not only a critical point, but in fact a global minimizer of the constrained variational problem for $E(u,v)$, then (1.7) is called a ground-state solution of (1.2). The same comments also apply to (1.4), except that the functional being varied in this case is $K(u,v)$. In this paper, our main concern is with ground-state solutions. For a discussion of what is currently known about bound-state solutions of (1.2) in general, see Section 2 below.

The terms "bound state" and "ground state" are traditional in the literature concerning the nonlinear Schrödinger equation

$$iu_t + u_{xx} = -u|u|^2. \tag{1.8}$$

Bound-state solutions of (1.8) are solutions of the form $u(x,t) = e^{i\omega t}h(x - ct)$, or equivalently minimizers of the Hamiltonian functional

$$\int_{-\infty}^{\infty} \left(|u_x|^2 - \frac{|u|^4}{2}\right)\ dx$$

subject to the constraints that $H(u)$ and $\int_{-\infty}^{\infty} u\overline{u}_x\, dx$ be held constant. It is easy to see that any bound-state solution of (1.8) must have a profile function of the form

$$h(x) = e^{i(cx/2+\theta)}\sqrt{2\sigma}\ \mathrm{sech}(\sqrt{\sigma}x + x_0)$$

where $\sigma = \omega - c^2/4 > 0$, and $x_0, \theta \in \mathbb{R}$. In fact, these bound states are actually ground states [12]. Since $|h(x)|$ decays monotonically to zero as x tends away from x_0 to ∞ or $-\infty$, bound-state solutions of (1.8) are often called solitary waves. By extension, the term "solitary wave" is often used to refer to bound-state solutions of equations which are related to (1.8), such as (1.2) or (1.4). This usage, however, is usually eschewed for bound states which are known not to have monotonic profiles, such as the excited bound states known to exist for generalizations of (1.8) to higher dimensions (see, e.g., [11]). Since, for system (1.2), we do not know in general whether the ground-state solutions we find have profiles which decay monotonically to zero away from a single extremum, we have here avoided calling them solitary waves.

Our main results are as follows. We prove below (see Theorem 4.5 and Corollary 5.2) that, for a certain range of values of q, equation (1.2) has for every $s > 0$ and $t \in \mathbb{R}$ a non-empty set of ground-state solutions (1.7) with profiles (h, g) satisfying $H(h) = s$ and $G(h, g) = t$. Moreover, for a given pair of values of s and t, the set $F_{s,t}$ of profiles of these solutions is stable, in the sense that if $(h, g) \in F_{s,t}$ and a slight perturbation of (h, g) is taken as initial data for (1.2), then the resulting solution of (1.2) can be said to have a profile which remains close to $F_{s,t}$ for all time (see Theorem 5.4).

Besides the main results, we also include an existence result for ground-state solutions of (1.2) which is valid for all $q > 0$ (Theorem 3.27) and an existence and stability result for ground-state solutions of (1.4) (Theorem 5.7). Concerning the latter result, we note that existence of bound-state solutions is obvious, since it is easy to explicitly find all solutions of the equations which result from substituting (1.7) into (1.4) (see Lemma 2.2 below). Also, the stability of these solutions has been proved by Laurençot in [24]. However, the method used by Laurençot did not establish whether these bound states were, in fact, ground states.

The results in the present paper are complementary to those contained in an earlier paper of one of us [4], where different techniques were used. In particular, it follows from the results of Section 3 of [4] that for every $q > 0$ we can find, for arbitrary $c > 0$ and arbitrary $\omega \in (c^2/4, \infty)$, a bound-state solution (1.7) of (1.2) such that $h(x) = e^{icx/2}f(x)$, where f is real-valued. Moreover, a stability result for certain sets of such bound states is proved when ω is near $c^2/4$. We also note that L. Chen [15] has proved the orbital stability of a two-parameter family of explicit bound-state solutions (see Section 2 below) in the special case $q = 2$. Finally, we mention the elegant proof in Ohta [33] of the stability of solitary-wave solutions of the Zakharov system,

$$\begin{cases} iu_t + u_{xx} = -uv \\ v_{tt} - v_{xx} = -(|u|^2)_{xx}, \end{cases} \tag{1.9}$$

by means of an argument which is related to the arguments used below in Section 4.

The proofs below follow the lines of many other proofs of existence and stability of solitary-wave solutions to dispersive equations which have appeared over the last couple of decades. The common elements in these proofs are the reduction of the stability problem to the problem of showing that minimizing sequences of a constrained variational problem are necessarily relatively compact, and the solution of this latter problem by the method of concentration compactness (see [13] for what may be the first example of such a stability proof).

In the present situation, however, application of the concentration compactness method is considerably complicated by the fact that, for a given choice of q in (1.2), we are interested in finding a true two-parameter family of bound-state solutions (parameterized by c and ω). In all the applications of the method to solitary waves which we are aware of, the variational problem has consisted of finding the extremum of a real-valued functional $E(f)$ subject to a single constraint of the form $Q(f) = \lambda$, where Q is another real-valued functional and $\lambda \in \mathbb{R}$ is a constant. This leads to a result concerning a one-parameter family of solitary waves. (In some cases, such as that of the nonlinear Schrödinger equation (1.8) or the Zakharov system (1.9), there at first appear to be two solitary-wave parameters, but it turns out that they are not independent.) Here, on the other hand, we are led to consider a variational problem in which there are not one but two real-valued constraint functions.

Now as was already noted in the original papers introducing the concentration compactness method (see, e.g., Section IV of [26]), the general outline of the method lends itself just as easily to problems in which there are more than one constraint function as to problems with a single constraint functional. But putting the method into practice requires proving the subadditivity of the variational problem with respect to the constraint parameters, and this turns out to be considerably more complicated in the case of two parameters. The task of proving the subadditivity of the relevant two-parameter variational problem will occupy us through most of Section 3.

The outline of this paper is as follows. In Section 2, we collect some basic facts concerning the properties of bound-state solutions of (1.2) and (1.4). Sections 3 and 4 contain the proof of the relative compactness of minimizing sequences for the variational problems which define ground-state solutions of (1.2) and (1.4). Finally, Section 5 discusses the existence and properties of ground-state solutions, including their stability properties.

Notation. We shall denote by $\widehat{f}$ the Fourier transform of f, defined as $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-i\xi x}\, dx$. For $1 \leq p \leq \infty$, we denote by $L^p = L^p(\mathbb{R})$ the space of all measurable functions f on $\mathbb{R}$ for which the norm $|f|_p$ is finite, where $|f|_p = \left(\int_{-\infty}^{\infty} |f|^p\, dx\right)^{1/p}$ for $1 \leq p < \infty$ and $|f|_\infty$ is the essential supremum of $|f|$ on $\mathbb{R}$. Whether we intend the functions in L^p to be real-valued or complex-valued will be clear from the context. For $s \geq 0$, we denote by $H^s_{\mathbb{C}} = H^s_{\mathbb{C}}(\mathbb{R})$ the Sobolev space of all complex-valued functions f in L^2 for which the norm

$$\|f\|_s = \left(\int_{-\infty}^{\infty} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2\, d\xi\right)^{1/2}$$

is finite. We will always view $H^s_{\mathbb{C}}$ as a vector space over the reals, with inner product given by $\langle f_1, f_2\rangle = \operatorname{Re}\int_{-\infty}^{\infty} (1 + |\xi|^2)^s \widehat{f_1}\overline{\widehat{f_2}}\, dx$. The space of all real-valued functions

f in $H^s_{\mathbb{C}}$ will be denoted simply by H^s. In particular, we use $\|f\|$ to denote the L^2 or H^0 norm of a function f. If I is an open interval in $\mathbb{R}$, we use $H^s(I)$ to denote the set of all functions f on I such that $f\eta \in H^s$ for every smooth function η with compact support in I. We define the space X to be the cartesian product $H^1_{\mathbb{C}} \times L^2$, and the space Y to be $H^1_{\mathbb{C}} \times H^1$, each provided with the product norm. Finally, if $T > 0$ and Z is any Banach space, we denote by $\mathcal{C}([0,T], Z)$ the Banach space of continuous maps $f : [0,T] \mapsto Z$, with norm given by $\|f\|_{\mathcal{C}([0,T],Z)} = \sup_{t\in[0,T]} \|f(t)\|_Z$. The letter C will frequently be used to denote various constants whose actual value is not important for our purposes.

2. Bound states

We record here some general results concerning bound-state solutions of (1.2) and related equations. We also include a list of explicit formulas for solutions in a few special cases, for purposes of comparison with the more general solutions we study in later sections.

Recall that a bound-state solution of (1.2) is, by definition, a solution of the form given in (1.7). In what follows, we further require that $h \in H^1_{\mathbb{C}}$ and $g \in H^1$. If we substitute (1.7) into (1.2), we can integrate the second of the resulting two equations, using the fact that $g \in H^1$ to evaluate the constant of integration. We see thus that $(u(x,t), v(x,t))$ is a bound-state solution of (1.2) if and only if h and g satisfy the equations

$$\begin{cases} h'' - \omega h - ich' = -hg \\ 2g'' - cg = -3qg^2 - |h|^2. \end{cases} \tag{2.1}$$

We can further simplify (2.1) by putting $h(x) = e^{icx/2}f(x)$, thus obtaining the system

$$\begin{cases} f'' - \sigma f = -fg \\ 2g'' - cg = -3qg^2 - |f|^2, \end{cases} \tag{2.2}$$

where $\sigma = \omega - \frac{c^2}{4}$. We can thus consider (2.2) to be the defining equations for bound-state solutions of (1.2).

Theorem 2.1. *Suppose $(f,g) \in Y$ is a solution of (2.2), in the sense of distributions. Then*

(i) $(f,g) \in H^\infty_{\mathbb{C}} \times H^\infty$.

(ii) if $c > 0$, then either f and g are both identically zero or $g(x) > 0$ for all $x \in \mathbb{R}$.

(iii) $f(x) = \varphi(x)e^{i\theta_0}$ for $x \in \mathbb{R}$, where θ_0 is a real constant and φ is real-valued.

(iv) if $\sigma > 0$ and $c > 0$, there exist constants $\epsilon_1, \epsilon_2 > 0$ such that $e^{\epsilon_1|x|}f(x)$ and $e^{\epsilon_2|x|}g(x)$ are in L^∞.

Proof. For any $s > 0$, define the function $K_s(x)$ by

$$K_s(x) = \frac{1}{2\sqrt{s}} e^{-\sqrt{s}|x|}.$$

Then $\widehat{K_s}(\xi) = (s+\xi^2)^{-1}$, so the operation of convolution with K_s takes $H^s_{\mathbb{C}}$ to $H^{s+2}_{\mathbb{C}}$, and is in fact the inverse of the operator $(s-\partial_{xx})$ in the sense that $(s-\partial_{xx})(K_s*f) = f$ for all $f \in H^s_{\mathbb{C}}$. Now we can rewrite (2.2) in the form

$$\begin{cases} f = K_{\sigma+a_1} * (fg + a_1 f) \\ g = K_{c/2+a_2} * \left(\left(\frac{3q}{2}\right) g^2 + \frac{1}{2}|f|^2 + a_2 g\right), \end{cases} \tag{2.3}$$

where a_1 and a_2 are real numbers chosen so that $\sigma + a_1 > 0$ and $c/2 + a_2 > 0$.

From (2.3), statement (i) follows by a standard bootstrap argument. Since f and g are in $H^1_{\mathbb{C}}$, and $H^1_{\mathbb{C}}$ is an algebra, then g^2, $|f|^2 = f\overline{f}$, and fg are also in $H^1_{\mathbb{C}}$. Hence (2.3) implies that f and g are in $H^3_{\mathbb{C}}$. But then g^2, $|f|^2$, and fg are in $H^3_{\mathbb{C}}$, so (2.3) implies that f and g are in $H^5_{\mathbb{C}}$, and so on.

To prove (ii), observe that if $c > 0$ then we can take $a_2 = 0$ in (2.3). But since $K_{c/2}$ is strictly positive on $\mathbb{R}$ and $g^2 + |f|^2$ is everywhere non-negative, it then follows from the second equation in (2.3) that if either f or g is non-zero on a set of positive measure, then $g(x) > 0$ everywhere.

For (iii), we first observe that by (i) and the standard uniqueness theory for ordinary differential equations, $f(x)$ and $f'(x)$ cannot both vanish at any point $x \in \mathbb{R}$. Moreover, if the zeros of f accumulate at any point $x \in \mathbb{R}$, then by Rolle's theorem, the zeros of $\mathrm{Re} f'$ and $\mathrm{Im} f'$ accumulate at x also, leading to the contradictory result that $f(x) = f'(x) = 0$. Therefore the zeros of f must be isolated.

Let x_1 and x_2 be any two consecutive zeros of f, where $x_1 < x_2$, and possibly $x_1 = -\infty$, or $x_2 = \infty$, or both. Then we can find infinitely differentiable functions r and θ on (x_1, x_2), with $r(x) > 0$ on (x_1, x_2) and $\lim_{x \to x_1^+} r(x) = \lim_{x \to x_2^-} r(x) = 0$, such that for all $x \in (x_1, x_2)$,

$$f(x) = r(x)e^{i\theta(x)}.$$

From the first equation in (2.2) we get

$$\begin{cases} r'' - \sigma r - r(\theta')^2 = -rg \\ 2r'\theta' + r\theta'' = 0. \end{cases} \tag{2.4}$$

Multiplying the second equation in (2.4) by $r(x)$ and integrating, we obtain

$$r^2(x)\theta'(x) = K$$

for all $x \in (x_1, x_2)$, where K is a constant. Now by (i), $|f'|^2 = (r')^2 + r^2(\theta')^2$ is bounded on $\mathbb{R}$, so $r^2(\theta')^2 = K^2/r^2$ is bounded on (x_1, x_2). But since $r \to 0$ as $x \to x_1$, this implies that $K = 0$ on (x_1, x_2). Hence θ is constant on (x_1, x_2).

The preceding argument shows that $f(x) = r(x)e^{i\theta(x)}$ on $\mathbb{R}$, where $\theta(x)$ is defined and constant on each of the intervals separating the zeros of $r(x)$. Now suppose that $x_0 \in \mathbb{R}$ is such that $r(x_0) = 0$, and define

$$\theta^- = \lim_{x \to x_0^-} \theta(x), \qquad t^- = \lim_{x \to x_0^-} r'(x),$$
$$\theta^+ = \lim_{x \to x_0^+} \theta(x), \qquad t^+ = \lim_{x \to x_0^+} r'(x).$$

Then $e^{i\theta^-}t^- = f'(x_0) = e^{i\theta^+}t^+$, and since $f'(x_0) \neq 0$, both t^- and t^+ are non-zero. Therefore $e^{i(\theta^+-\theta^-)} = t^-/t^+ \in \mathbb{R}$, from which it follows that $e^{i(\theta^+-\theta^-)}$ is either 1 or -1. Hence we can arrange that $f(x) = \varphi(x)e^{i\theta_0}$ on both sides of x_0, where $\varphi(x)$ is real-valued, by taking $\theta_0 = \theta^-$ and defining $\varphi(x) = r(x)$ for x to the left of x_0 and $\varphi(x) = r(x)e^{i(\theta^+-\theta^-)}$ to the right of x_0. Stepping through the intervals between zeros of $r(x)$ one at a time, both rightward and leftward from x_0, and iterating this procedure, one obtains the desired result.

To prove (iv), we borrow an argument from the proof of Theorem 8.1.1(iv) of [12]. For each $\epsilon > 0$ and $\eta > 0$ define a function ζ by $\zeta(x) = e^{\epsilon|x|/(1+\eta|x|)}$. Multiply the first equation in (2.2) by $\zeta\overline{f}$ and add the result to its complex conjugate to get

$$\operatorname{Re}\int_{-\infty}^{\infty} f'(\zeta\overline{f})'\,dx + \sigma\int_{-\infty}^{\infty}\zeta|f|^2\,dx = \int_{-\infty}^{\infty}\zeta g|f|^2\,dx.$$

Since $\zeta' \leq \epsilon\zeta$, we can deduce that

$$\sigma\int_{-\infty}^{\infty}\zeta|f|^2\,dx \leq \int_{-\infty}^{\infty}\zeta g|f|^2\,dx - \int_{-\infty}^{\infty}\zeta|f'|^2\,dx + \epsilon\int_{-\infty}^{\infty}\zeta|ff'|\,dx. \tag{2.5}$$

Now using the Cauchy-Schwarz inequality with ϵ chosen to be sufficiently small, we deduce from (2.5) that

$$\int_{-\infty}^{\infty}\zeta|f|^2\,dx \leq C_\epsilon\int_{-\infty}^{\infty}\zeta g|f|^2\,dx, \tag{2.6}$$

where C_ϵ does not depend on η. Since $g \in H^1$, we can find $R > 0$ such that $|g(x)| \leq 1/(2C_\epsilon)$ for $|x| \geq R$. It then follows from (2.6) that

$$\int_{-\infty}^{\infty}\zeta|f|^2\,dx \leq 2C_\epsilon\int_{|x|\leq R} e^{\epsilon|x|}g(x)|f(x)|^2\,dx,$$

and taking $\eta \to 0$ gives

$$\int_{-\infty}^{\infty} e^{\epsilon|x|}|f(x)|^2\,dx < \infty. \tag{2.7}$$

Now since $f \in H^1$, then $f(x) \to 0$ as $|x| \to \infty$ and f is uniformly Lipschitz on $\mathbb{R}$. From these two properties of f and (2.7) it follows easily that $e^{\epsilon_1|x|}f(x)$ is bounded on $\mathbb{R}$ for some $\epsilon_1 \in (0,\epsilon)$ (for details, see the proof of Theorem 8.1.7(iv) of [12]).

The decay estimate for g is obtained in the same way as that for f. Multiplying the second equation in (2.2) by ζg leads, as above, to the estimate

$$\int_{-\infty}^{\infty}\zeta g^2\,dx \leq C_\epsilon\int_{-\infty}^{\infty}\zeta\left(g^3 + |f|^2 g\right)\,dx.$$

Choosing $\epsilon < 2\epsilon_1$, and using the decay result just proved for f, we find as before that $\int_{-\infty}^{\infty}\zeta g^2\,dx$ can be bounded by a constant which is independent of η. Taking $\eta \to 0$ allows us to conclude that

$$\int_{-\infty}^{\infty} e^{\epsilon|x|}|g(x)|^2\,dx < \infty,$$

and from here the proof proceeds as it did for $f(x)$. □

Funakoshi and Oikawa, in [17], list the following explicit one-parameter families of bound-state solutions to (1.2) and (1.3). For $q \leq 2/3$, define

$$\begin{cases} f(x) = \pm 6B^2\sqrt{2-3q}\ \operatorname{sech}^2(Bx) \\ g(x) = 6B^2\ \operatorname{sech}^2(Bx), \end{cases} \tag{2.8}$$

where $B > 0$ is arbitrary. Then (f, g) satisfy (2.2) with $\sigma = 4B^2$ and $c = 8B^2$. If, on the other hand, $q \geq 2/3$, then we have that

$$\begin{cases} f(x) = \pm 6B^2\sqrt{3q-2}\ \operatorname{sech}(Bx)\tanh(Bx) \\ g(x) = 6B^2\ \operatorname{sech}^2(Bx) \end{cases} \tag{2.9}$$

is a solution of (2.2) with $\sigma = B^2$ and $c = 2B^2(9q-2)$. When $q = 2/3$, of course, these solutions coincide with the obvious solution given by $f = 0$ and $g = (4B^2/q)\ \operatorname{sech}^2(Bx)$, which satisfies (2.2) with $c = 8B^2$ for all $q \neq 0$.

In [15], L. Chen considered (1.2) in the special case when $q = 2$, and found a two-parameter family of explicit solutions, given by

$$\begin{cases} f(x) = \pm\sqrt{2B^2(c-8B^2)}\ \operatorname{sech}(Bx) \\ g(x) = 2B^2\ \operatorname{sech}^2(Bx), \end{cases} \tag{2.10}$$

where $B^2 = \sigma$, and $c > 0$ and $\sigma \in (0, c/8)$ are arbitrary. Then, using the stability theory of [18], he went on to show that if $h(x) = e^{icx/2}f(x)$, $\omega = \sigma + c^2/4$, and (u, v) is the bound-state solution of (1.2) defined by (2.10) and (1.7), then (u, v) is orbitally stable provided $c \leq 1$ and $\sigma \in (0, c/12)$ (see Theorem 2 of [15]). Here, orbital stability of (u, v) means that if F, the orbit of (f, g), is defined as the set of all $(\tilde{f}, \tilde{g}) \in Y$ such that $\tilde{f}(x) = e^{i\theta_0}f(x + x_0)$ and $\tilde{g}(x) = g(x + x_0)$ for some $\theta_0, x_0 \in \mathbb{R}$, then F is stable in the sense of Theorem 5.2 below.

In Theorem 5.1 below, it is shown that if (f, g) is a solution of (2.2) corresponding to a ground-state solution of (1.2), then up to a multiplicative constant of absolute value one, f is a positive function on $\mathbb{R}$. Therefore the bound state given by (2.9) is not a ground state. In fact, in the case $q = 2$ it is not hard to show (see remark 3.18 below) that there is, up to translation and phase shift, a unique ground-state solution of (2.2), and that this solution is given by (2.10). We do not know, however, whether ground states are unique for $q \neq 2$.

In later sections, we will need the following uniqueness results for certain equations related to (2.2).

Lemma 2.2. *Suppose $(f, g) \in X$ is a non-zero solution of the equations*

$$\begin{cases} f'' + fg = \lambda f \\ |f|^2 = \mu g, \end{cases} \tag{2.11}$$

where $\lambda, \mu \in \mathbb{R}$. Then $\lambda > 0$ and $\mu > 0$, and $f(x) = e^{i\theta_0}f_1(x + x_0)$ and $g(x) = g_1(x + x_0)$, where $\theta_0, x_0 \in \mathbb{R}$ and

$$\begin{cases} f_1(x) = \sqrt{2\lambda\mu}\,\operatorname{sech}(\sqrt{\lambda}x) \\ g_1(x) = 2\lambda\ \operatorname{sech}^2(\sqrt{\lambda}x). \end{cases} \tag{2.12}$$

Lemma 2.3. *Suppose $g \in H^1$ is a non-zero solution of the equation*

$$-g'' - \frac{3q}{2} g^2 = \kappa g, \tag{2.13}$$

where $\kappa \in \mathbb{R}$. Then $\kappa > 0$ and $g = g_2(x + x_0)$, where $x_0 \in \mathbb{R}$ and

$$g_2(x) = \frac{\kappa}{q} \operatorname{sech}^2 \left(\frac{\sqrt{\kappa}}{2} x \right). \tag{2.14}$$

To prove these well-known results, one begins by using a bootstrap argument to establish that any solution must in fact be infinitely differentiable. Equation (2.13) can then be integrated twice (after first multiplying by g'), to yield (2.14). For equation (2.11), we can argue as in the proof of Theorem 2.1(iii) to show that $f(x) = e^{i\theta_0 x}\varphi(x)$, where φ is real-valued, and then eliminate g to obtain a single equation for φ, which may be solved by integrating twice. We omit the details.

3. The reduced variational problem

In this section we consider the problem of finding

$$I(s,t) = \inf \left\{ E(f,g) : (f,g) \in Y,\ \|f\|^2 = s, \text{ and } \|g\|^2 = t \right\}, \tag{3.1}$$

where $s, t > 0$. Our approach will be to split the functional E into two parts and consider the variational problem associated with each part. Define $K : X \to \mathbb{R}$ by

$$K(f,g) = \int_{-\infty}^{\infty} \left(|f'|^2 - g|f|^2 \right) dx,$$

and $J : H^1 \to \mathbb{R}$ by

$$J(g) = \int_{-\infty}^{\infty} \left((g')^2 - qg^3 \right) \, dx.$$

Then

$$E(f,g) = K(f,g) + J(g).$$

Hence, if we define $M : H^1 \to \mathbb{R}$ by

$$M(g) = \inf \left\{ K(f,g) : f \in H^1_{\mathbb{C}} \text{ and } \|f\| = 1 \right\}, \tag{3.2}$$

then

$$I(s,t) = \inf \left\{ sM(g) + J(g) : g \in H^1 \text{ and } \|g\|^2 = t \right\}. \tag{3.3}$$

This expression for $I(s,t)$ suggests analyzing the subsidiary variational problems defined by

$$\begin{aligned} I_1(s,t) &= \inf \left\{ K(f,g) : (f,g) \in X,\ \|f\|^2 = s, \text{ and } \|g\|^2 = t \right\} \\ &= \inf \left\{ sM(g) : \ g \in H^1 \text{ and } \|g\|^2 = t \right\} \end{aligned} \tag{3.4}$$

and

$$I_2(t) = \inf \left\{ J(g) : g \in H^1 \text{ and } \|g\|^2 = t \right\}. \tag{3.5}$$

Lemma 3.1. *If $(f,g) \in X$, then $(|f|,g) \in X$ also, and $K(|f|,g) \le K(f,g)$.*

Proof. What has to be proved is that if $f \in H^1_{\mathbb{C}}$, then $F(x) = |f(x)|$ is in H^1, with $\|F\|_1 \le \|f\|_1$. We do not prove this elementary fact here, but remark that a proof can be given which, by working with $\widehat{f}$ and $\widehat{F}$ instead of f and F, avoids the annoying question of the differentiability of F at points where $F = 0$. Such a proof is easily constructed by adapting the proof of Lemma 3.4 in [1]. □

Lemma 3.2. *For all $s, t \ge 0$, $I_1(s,t)$ and $I_2(t)$ are finite.*

Proof. Let $(f,g) \in X$ with $\|f\|^2 = s$ and $\|g\|^2 = t$. Then from the Cauchy-Schwartz inequality and the Sobolev embedding theorem we have

$$\left| \int_{-\infty}^{\infty} g|f|^2 \, dx \right| \le C\|f\|_1 \|f\| \|g\| \le \int_{-\infty}^{\infty} |f'|^2 \, dx + Cs(1+t)$$

and

$$\left| \int_{-\infty}^{\infty} g^3 \, dx \right| \le C\|g\|_1 \|g\|^2 \le \int_{-\infty}^{\infty} (g')^2 \, dx + Cs^2.$$

Hence $I_1(s,t) \ge -Cs(1+t) > -\infty$ and $I_2(t) \ge -Cs^2 > -\infty$. □

Lemma 3.3. *For all $s, t > 0$ we have $I_1(s,t) < 0$ and $I_2(t) < 0$. Also, $I_1(s,0) = 0$ for all $s \ge 0$, $I_1(0,t) = 0$ for all $t \ge 0$, and $I_2(0) = 0$.*

Proof. When $s, t > 0$ we can choose $(f,g) \in X$ such that $\|f\|^2 = s$, $\|g\|^2 = t$, $\int_{-\infty}^{\infty} g|f|^2 \, dx > 0$, and $\int g^3 \, dx > 0$. Then for each $\theta > 0$, the functions $f_\theta(x) = \theta^{1/2} f(\theta x)$ and $g_\theta(x) = \theta^{1/2} g(\theta x)$ satisfy $\|f_\theta\|^2 = s$, $\|g_\theta\|^2 = t$,

$$K(f_\theta, g_\theta) = \theta^2 \int_{-\infty}^{\infty} |f'|^2 \, dx - \theta^{1/2} \int_{-\infty}^{\infty} g|f|^2 \, dx,$$

and

$$J(g_\theta) = \theta^2 \int_{-\infty}^{\infty} (g')^2 \, dx - \theta^{1/2} \int_{-\infty}^{\infty} g^3 \, dx.$$

Hence, by taking θ sufficiently small, we get $K(f_\theta, g_\theta) < 0$ and $J(g_\theta) < 0$, proving that $I_1(s,t) < 0$ and $I_2(t) < 0$.

If $s \ge 0$, then choosing any $f \in H^1$ with $\|f\| = s$ and defining f_θ as in the preceding paragraph, we get

$$K(f_\theta, 0) = \theta^2 \int_{-\infty}^{\infty} |f'|^2 \, dx \ge I_1(s,0) \ge 0.$$

Then by letting θ tend to zero we see that $I_1(s,0) = 0$.

Finally, the equalities $I_1(0,t) = 0$ and $I_2(0) = 0$ are obvious. □

Lemma 3.4. *Suppose $\sigma > 0$, and define a map $g \to g^*$ from H^1 onto H^1 by*

$$g^*(x) = \sigma^{2/3} g(\sigma^{1/3} x).$$

Then for each $g \in H^1$,

$$M(g^*) = \sigma^{2/3} M(g) \tag{3.6}$$

and

$$J(g^*) = \sigma^{5/3} J(g). \tag{3.7}$$

Proof. A simple change of variables in the integral proves (3.7). To prove (3.6), for each $f \in H^1_{\mathbb{C}}$ such that $\|f\| = 1$, define $\tilde{f}$ by

$$\tilde{f}(x) = \sigma^{1/6} f(\sigma^{1/3} x).$$

Then $\|\tilde{f}\| = 1$ and $K(\tilde{f}, g^*) = \sigma^{2/3} K(f,g)$, whence (3.6) follows by taking infima on both sides. □

Lemma 3.5. *For all $s, t \geq 0$, we have*

$$I_1(s,t) = st^{2/3} I_1(1,1) \tag{3.8}$$

and

$$I_2(t) = t^{5/3} I_2(1) \tag{3.9}$$

Proof. We may assume $s, t > 0$. Let $(f,g) \in X$ be such that $\|f\|^2 = s$ and $\|g\|^2 = t$, and let $\tilde{f}$ and g^* be as defined in Lemma 3.4 and its proof, with $\sigma = t^{-1}$. Define $z = s^{-1/2} \tilde{f}$. Then $\|z\|^2 = 1$, $\|g^*\|^2 = 1$,

$$K(f,g) = st^{2/3} K(z, g^*), \tag{3.10}$$

and

$$J(g) = t^{5/3} J(g^*). \tag{3.11}$$

The equality (3.8) follows by taking the infimum of both sides of (3.10) with respect to f and g, while (3.9) follows by taking the infimum of both sides of (3.11) with respect to g. □

Lemma 3.6. *Suppose $s_1, t_1, s_2, t_2 > 0$. If $t_1/t_2 = s_1/s_2 = \sigma$, then*

$$I(s_1, t_1) = \sigma^{5/3} I(s_2, t_2).$$

Proof. For $g \in H^1$, let g^* be as defined in Lemma 3.4. Then

$$\begin{aligned} I(s_1, t_1) &= \inf\{s_1 M(g^*) + J(g^*) : g^* \in H^1 \text{ and } \|g^*\|^2 = t_1\} \\ &= \inf\{\sigma^{5/3} \left(s_2 M(g) + J(g)\right) : g \in H^1 \text{ and } \|g\|^2 = t_2\} \\ &= \sigma^{5/3} I(s_2, t_2). \end{aligned}$$

□

Lemma 3.7. *Let $s_1, s_2, t_1, t_2 \geq 0$, and suppose that $s_1 + s_2 > 0$, $t_1 + t_2 > 0$, $s_1 + t_1 > 0$, and $s_2 + t_2 > 0$. Then*

$$I_1(s_1 + s_2, t_1 + t_2) < I_1(s_1, t_1) + I_1(s_2, t_2). \tag{3.12}$$

Also, if $t_1, t_2 > 0$ then

$$I_2(t_1 + t_2) < I_2(t_1) + I_2(t_2). \tag{3.13}$$

Proof. To prove (3.12), we consider three cases: when $s_1 = 0$, when $t_1 = 0$, and when neither s_1 nor t_1 is 0. In the first case, we must have $s_2 > 0$ and $t_1 > 0$, so

$$s_2(t_1 + t_2)^{2/3} > s_2 t_2^{2/3}.$$

Since $I_1(1, 1) < 0$ and $I_1(s_1, t_1) = 0$ by Lemma 3.3, multiplying both sides by $I(1, 1)$ and using Lemma 3.5 gives the desired inequality. Similarly, in the second case, we must have $s_1 > 0$ and $t_2 > 0$, so

$$(s_1 + s_2)(t_1 + t_2)^{2/3} > s_1 t_1^{2/3} + s_2 t_2^{2/3},$$

and again multiplying by $I_1(1, 1)$ gives the desired inequality. Finally, in the third case, when $s_1 > 0$ and $t_1 > 0$, we must have either $s_2 > 0$ or $t_2 > 0$. If $s_2 > 0$, then we write

$$\begin{aligned}(s_1 + s_2)(t_1 + t_2)^{2/3} &= s_1(t_1 + t_2)^{2/3} + s_2(t_1 + t_2)^{2/3} \\ &> s_1(t_1 + t_2)^{2/3} + s_2 t_2^{2/3} \geq s_1 t_1^{2/3} + s_2 t_2^{2/3}.\end{aligned}$$

If $t_2 > 0$, we can write the same string of inequalities, with the penultimate expression replaced by $s_1 t_1^{2/3} + s_2(t_1 + t_2)^{2/3}$. In either case, we have established that

$$(s_1 + s_2)(t_1 + t_2)^{2/3} > s_1 t_1^{2/3} + s_2 t_2^{2/3},$$

which, when multiplied by $I_1(1, 1) < 0$, gives the desired result.

To prove (3.13), we merely observe that

$$(t_1 + t_2)^{5/3} > t_1^{5/3} + t_2^{5/3}$$

for $t_1, t_2 > 0$, and apply Lemma 3.3 and Lemma 3.5. □

The following result, which we state here without proof, is taken from Lemma 2.4 of [14]. For a proof, see Lemma I.1 of [26].

Lemma 3.8. *Suppose $p, r \in [1, \infty)$, $\{f_n\}$ is a bounded sequence in L^r, and $\{f_n'\}$ is bounded in L^p. If, for some $\omega > 0$,*

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}} \int_{y-\omega}^{y+\omega} |f_n|^r \, dx = 0$$

then for every $s > r$,

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} |f_n|^s \, dx = 0.$$

We will now prove the existence of minimizing pairs for problems (3.4) and (3.5). Actually, we accomplish somewhat more: using the method of concentration compactness [25,26], we show that in fact every minimizing sequence for these variational problems has a subsequence which converges, after suitable translations, to a solution of the problem. From this property of minimizing sequences there easily follow stability results for the evolution equations (1.2) and (1.4); see Theorems 5.4 and 5.7 below.

Let us first consider minimizing sequences for (3.4), which are by definition sequences $\{(f_n, g_n)\}$ in X satisfying

$$\lim_{n\to\infty} \|f_n\|^2 = s, \qquad \lim_{n\to\infty} \|g_n\|^2 = t, \qquad \text{and} \qquad \lim_{n\to\infty} K(f_n, g_n) = I_1(s,t).$$

(Note that we do not require the elements (f_n, g_n) of a minimizing sequence to satisfy exactly the constraints in (3.4). This convention will be useful later, in the proof of Theorem 5.4.) To each such sequence we associate a sequence of nondecreasing functions $Q_n(\omega)$, defined for $\omega > 0$ by

$$Q_n(\omega) = \sup_{y\in\mathbb{R}} \int_{y-\omega}^{y+\omega} \left(|f_n|^2(x) + g_n^2(x)\right)\, dx.$$

Since $\|f_n\|$ and $\|g_n\|$ remain bounded, then $\{Q_n\}$ comprises a uniformly bounded sequence of nondecreasing functions on $[0,\infty)$. A standard argument then implies that $\{Q_n\}$ must have a subsequence, which we denote again by $\{Q_n\}$, that converges pointwise and uniformly on compact sets to a nondecreasing limit function on $[0,\infty)$. Let Q be this limit function, and define

$$\alpha = \lim_{\omega\to\infty} Q(\omega). \tag{3.14}$$

From the assumption that $\|f_n\|^2 + \|g_n\|^2 \to s+t$ it follows that $0 \leqq \alpha \leqq s+t$. The concentration-compactness method distinguishes three cases: $\alpha = s+t$, called the case of *compactness*; $\alpha = 0$, called the case of *vanishing*; and $0 < \alpha < s+t$, called the case of *dichotomy*. Our goal is to show that for minimizing sequences of (3.4), only the case of compactness can occur. It will follow, by a standard argument, that every minimizing sequence is relatively compact, after suitable translations (cf. Theorem 3.12 below). Later, we will show that this compactness property is also enjoyed by problem (3.1).

Lemma 3.9. *Suppose $s,t \geq 0$. If $\{(f_n, g_n)\}$ is a minimizing sequence for $I_1(s,t)$, then $\{(f_n, g_n)\}$ is bounded in X.*

Proof. From standard Sobolev embedding and interpolation theorems we have

$$\left| \int_{-\infty}^{\infty} g_n |f_n|^2\, dx \right| \leq |f_n|_4^2 \|g_n\| \leq C \|f_n\|_1^{1/2} \|f_n\|^{3/2} \|g_n\|.$$

But for a minimizing sequence, $\|f_n\|$ and $\|g_n\|$ stay bounded, so it follows that

$$\left| \int_{-\infty}^{\infty} g_n |f_n|^2\, dx \right| \leq C \|f_n\|_1^{1/2},$$

where C is independent of n. Hence, since $\{K(f_n, g_n)\}$ is a bounded sequence, we obtain

$$\|f_n\|_1^2 = K(f_n, g_n) + \int_{-\infty}^{\infty} g_n |f_n|^2\, dx + \|f_n\|^2 \leq C(1 + \|f_n\|_1^{1/2}),$$

from which it follows that $\|f_n\|_1$ is bounded. Therefore

$$\|(f_n, g_n)\|_X^2 = \|f_n\|_1^2 + \|g_n\|^2 \leq C,$$

and we are done. □

Lemma 3.10. *Suppose $s, t > 0$, and let $\{(f_n, g_n)\}$ be any minimizing sequence for $I_1(s,t)$. Let α be as defined in* (3.14). *Then there exist numbers $s_1 \in [0,s]$ and $t_1 \in [0,t]$ such that*

$$s_1 + t_1 = \alpha \tag{3.15}$$

and

$$I_1(s_1,t_1) + I_1(s - s_1, t - t_1) \le I_1(s,t). \tag{3.16}$$

Proof. Let ϵ be an arbitrary positive number. From the definition of α it follows that for ω sufficiently large we have $\alpha - \epsilon < Q(\omega) \leqq Q(2\omega) \leqq \alpha$. By taking ω larger if necessary, we may also assume that $1/\omega < \epsilon$. Now according to the definition of Q we can choose N so large that, for every $n \ge N$,

$$\alpha - \epsilon < Q_n(\omega) \le Q_n(2\omega) \leqq \alpha + \epsilon \tag{3.17}$$

Hence for each $n \geqq N$ we can find y_n such that

$$\int_{y_n-\omega}^{y_n+\omega} \left(|f_n|^2 + g_n^2\right)\, dx > \alpha - \epsilon \quad \text{and} \quad \int_{y_n-2\omega}^{y_n+2\omega} \left(|f_n|^2 + g_n^2\right)\, dx < \alpha + \epsilon. \tag{3.18}$$

Now choose smooth functions p and r on $\mathbb{R}$ such that $p(x) = 1$ for $x \in [-1,1]$, $p(x) = 0$ for $x \notin [-2,2]$, $r(x) = 1$ for $x \notin [-2,2]$, $r(x) = 0$ for $x \in [-1,1]$, and $p^2(x) + r^2(x) = 1$ for all $x \in \mathbb{R}$. Define $p_\omega(x) = p(\frac{x}{\omega})$ and $r_\omega(x) = r(\frac{x}{\omega})$, and let

$$(\varphi_n(x), h_n(x)) = (p_\omega(x - y_n) f_n(x), p_\omega(x - y_n) g_n(x))$$

and

$$(l_n(x), j_n(x)) = (r_\omega(x - y_n) f_n(x), r_\omega(x - y_n) g_n(x)).$$

From Lemma 3.9 it follows that the sequences $\{\varphi_n\}$, $\{h_n\}$, $\{l_n\}$, and $\{j_n\}$ are bounded in L^2. So by passing to subsequences, we may assume that there exist $s_1 \in [0,s]$ and $t_1 \in [0,t]$ such that $\int_{-\infty}^{\infty} |\varphi_n|^2\, dx \to s_1$ and $\int_{-\infty}^{\infty} h_n^2\, dx \to t_1$, whence it follows also that $\int_{-\infty}^{\infty} |l_n|^2\, dx \to s - s_1$ and $\int_{-\infty}^{\infty} j_n^2\, dx \to t - t_1$. Now

$$s_1 + t_1 = \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|\varphi_n|^2 + h_n^2\right)\, dx = \lim_{n\to\infty} \int_{-\infty}^{\infty} p_\omega^2 \left(|f_n|^2 + g_n^2\right)\, dx.$$

Here and below we have suppressed the arguments of p_ω and r_ω for brevity of notation. From (3.18) it follows that, for every $n \in \mathbb{N}$,

$$\alpha - \epsilon < \int_{-\infty}^{\infty} p_\omega^2 \left(|f_n|^2 + g_n^2\right)\, dx < \alpha + \epsilon.$$

Hence

$$|(s_1 + t_1) - \alpha| < \epsilon.$$

Next observe that

$$|p_\omega'|_\infty + |r_\omega'|_\infty \leqq \frac{1}{\omega}\left(|p'|_\infty + |r'|_\infty\right) \le \frac{C}{\omega},$$

and, by Lemma 3.9, $\|f_n\|_1 \le C$, where C denotes constants which are independent of ω and n. Hence

$$K(\varphi_n, h_n) \le \int_{-\infty}^{\infty} \left(p_\omega^2 |f_n'|^2 - p_\omega^2 g_n |f_n|^2\right)\, dx + \int_{-\infty}^{\infty} (p_\omega^2 - p_\omega^3) g_n |f_n|^2\, dx + \frac{C}{\omega} \quad (3.19)$$

and

$$K(l_n, j_n) \le \int_{-\infty}^{\infty} \left(r_\omega^2 |f_n'|^2 - r_\omega^2 g_n |f_n|^2\right)\, dx + \int_{-\infty}^{\infty} (r_\omega^2 - r_\omega^3) g_n |f_n|^2\, dx + \frac{C}{\omega}. \quad (3.20)$$

On the other hand, from (3.18) we get

$$\left| \int_{-\infty}^{\infty} \left((p_\omega^2 - p_\omega^3) + (r_\omega^2 - r_\omega^3)\right) g_n |f_n|^2\, dx \right| \le 2|f_n|_\infty \int_{\omega \leqq |x - y_n| \leqq 2\omega} \left(|f_n|^2 + g_n^2\right)\, dx \le C\epsilon.$$

Therefore, adding (3.19) and (3.20) and using $p_\omega^2 + r_\omega^2 = 1$, we get

$$K(\varphi_n, h_n) + K(l_n, j_n) \le K(f_n, g_n) + C\left(\epsilon + \frac{1}{\omega}\right) \le K(f_n, g_n) + C\epsilon. \quad (3.21)$$

For any given value of ϵ, each of the terms in (3.21) is bounded independently of n, so by passing to subsequences we may assume that $K(\varphi_n, h_n) \to K_1$ and $K(l_n, j_n) \to K_2$, where

$$K_1 + K_2 \le I_1(s,t) + C\epsilon.$$

Combining the results of the preceding paragraphs, and recalling that ϵ can be taken arbitrarily small and ω arbitrarily large, we see that for every $k \in \mathbb{N}$, we can find sequences $\{(\varphi_n^{(k)}, h_n^{(k)})\}$ and $\{(l_n^{(k)}, j_n^{(k)})\}$ in X such that $\|\varphi_n^{(k)}\|^2 \to s_1(k)$, $\|h_n^{(k)}\|^2 \to t_1(k)$, $\|l_n^{(k)}\|^2 \to s - s_1(k)$, $\|j_n^{(k)}\|^2 \to t - t_1(k)$, $K(\varphi_n^{(k)}, h_n^{(k)}) \to K_1(k)$, and $K(l_n^{(k)}, j_n^{(k)}) \to K_2(k)$, where $s_1(k) \in [0,s]$, $t_1(k) \in [0,t]$,

$$|s_1(k) + t_1(k) - \alpha| \le \frac{1}{k}, \quad (3.22)$$

and

$$K_1(k) + K_2(k) \le I_1(s,t) + \frac{1}{k}. \quad (3.23)$$

By passing to subsequences we may assume that $s_1(k)$, $t_1(k)$, $K_1(k)$, and $K_2(k)$ converge to numbers $s_1 \in [0,s]$, $t_1 \in [0,t]$, K_1, and K_2. Moreover, by redefining (φ_n, g_n) and (h_n, j_n) as the diagonal subsequences $(\varphi_n, g_n) = (\varphi_n^{(n)}, g_n^{(n)})$ and $(h_n, j_n) = (h_n^{(n)}, j_n^{(n)})$, we may assume that $\|\varphi_n\|^2 \to s_1$, $\|h_n\|^2 \to t_1$, $\|l_n\|^2 \to s - s_1$, $\|j_n\|^2 \to t - t_1$, $K(\varphi_n, g_n) \to K_1$, and $K(h_n, j_n) \to K_2$.

Now letting $k \to \infty$ in (3.22) gives (3.15), and similarly (3.23) will imply (3.16), provided we can show that

$$K_1 \ge I_1(s_1, t_1) \quad (3.24)$$

and

$$K_2 \ge I_1(s - s_1, t - t_1). \quad (3.25)$$

To prove (3.24), we consider three cases: (i) $s_1 > 0$ and $t_1 > 0$, (ii) $s_1 = 0$, and (iii) $t_1 = 0$. In case (i), for n sufficiently large we have $\|\varphi_n\| > 0$ and $\|h_n\| > 0$, so we may define $\beta_n = \sqrt{s_1}/\|\varphi_n\|$ and $\theta_n = \sqrt{t_1}/\|h_n\|$. Then $\|\beta_n\varphi_n\|^2 = s_1$ and $\|\theta_n h_n\|^2 = t_1$, so

$$K(\beta_n\varphi_n, \theta_n h_n) \geq I_1(s_1, t_1).$$

But since β_n and θ_n approach 1 as $n \to \infty$, we have $K(\beta_n\varphi_n, \theta_n h_n) \to K_1$, from which (3.24) follows. In case $s_1 = 0$, we have $\|\varphi_n\| \to 0$, so

$$\left|\int_{-\infty}^{\infty} h_n|\varphi_n|^2\ dx\right| \leq \|\varphi_n\|_1\|\varphi_n\|\|h_n\| \to 0, \tag{3.26}$$

whence

$$K_1 = \lim_{n\to\infty} K(\varphi_n, h_n) = \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|\varphi_n'|^2 - h_n|\varphi_n|^2\right)\ dx \geq 0. \tag{3.27}$$

Since $I_1(s_1, t_1) = I_1(0, t_1) = 0$, this proves (3.24) in case (ii). Finally, if $t_1 = 0$, then $\|h_n\| \to 0$, so (3.26) and (3.27) again hold, which proves (3.24) in this case since $I_1(s_1, 0) = 0$. Therefore (3.24) has been proved in all cases. The proof of (3.25) is similar, with $s - s_1$ and $t - t_1$ playing the roles of s_1 and t_1. □

Lemma 3.11. *Suppose $s, t > 0$, and let $\{(f_n, g_n)\}$ be any minimizing sequence for $I_1(s,t)$. If α is as defined in (3.14), then $\alpha = s + t$.*

Proof. First we show that $\alpha \neq 0$. If $\alpha = 0$, then $\sup_{y\in\mathbb{R}} \int_{y-\omega}^{y+\omega} |f_n|^q\ dx \to 0$ for every $\omega > 0$, so Lemma 3.8 implies that $f_n \to 0$ in L^4. But then, since

$$\left|\int_{-\infty}^{\infty} g_n|f_n|^2\ dx\right| \leq |f_n|_4^{1/2}\|g_n\|$$

and $\|g_n\|$ stays bounded, we have that $\int_{-\infty}^{\infty} g_n|f_n|^2\ dx \to 0$ as $n \to \infty$. Therefore,

$$I_1(s,t) = \lim_{n\to\infty} K(f_n, g_n) \geqq \liminf_{n\to\infty} \int_{-\infty}^{\infty} |f_n'|^2\ dx \geqq 0,$$

which contradicts Lemma 3.3.

It remains then to show that α cannot lie in $(0, s+t)$. Suppose to the contrary that $0 < \alpha < s+t$. Let s_1 and t_1 be as defined in Lemma 3.10, and let $s_2 = s - s_1$, $t_2 = t - t_1$. Then (3.15) implies both that $s_1 + t_1 = \alpha > 0$ and $s_2 + t_2 = (s+t) - \alpha > 0$. Since $s_1 + s_2 = s > 0$ and $t_1 + t_2 = t > 0$, we conclude from Lemma 3.7 that (3.12) holds. But this contradicts (3.16). □

Theorem 3.12. *Let $s, t > 0$, and let $\{(f_n, g_n)\}$ be any minimizing sequence for $I_1(s,t)$. Then there is a subsequence $\{(f_{n_k}, g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that*

$$(f_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k))$$

converges strongly in X to some (f,g). The pair (f,g) is a minimizer for $I_1(s,t)$; i.e, $\|f\|^2 = s$, $\|g\|^2 = t$, and $K(f,g) = sM(g) = I_1(s,t)$.

Proof. The proof is a variation on that of the fundamental Lemma I.1(i) of [25]. For any minimizing sequence $\{(f_n, g_n)\}$ of $I_1(s,t)$, define α as in (3.14), and let $\{(f_n, g_n)\}$ continue to denote the subsequence associated with α. From Lemma 3.11 we have that $\alpha = s + t$. Hence there exists ω_0 such that for n sufficiently large, $Q_n(\omega_0) > \frac{s+t}{2}$. For such n, we choose y_n such that

$$\int_{y_n-\omega_0}^{y_n+\omega_0} \left(|f_n|^2 + g_n^2\right)\ dx > \frac{s+t}{2}.$$

Now let σ be an arbitrary number in the interval $(\frac{s+t}{2}, s+t)$. Then we can find ω_1 such that for n sufficiently large, $Q_n(\omega_1) > \sigma$, and so we can choose $\tilde{y}_n$ such that

$$\int_{\tilde{y}_n-\omega_1}^{\tilde{y}_n+\omega_1} \left(|f_n|^2 + g_n^2\right)\ dx > \sigma.$$

Since $\int_{-\infty}^{\infty} \left(|f_n|^2 + g_n^2\right)\ dx \to s + t$ as $n \to \infty$, it follows that for large n, the intervals $[\tilde{y}_n - \omega_1, \tilde{y}_n + \omega_1]$ and $[y_n - \omega_0, y_n + \omega_0]$ must overlap. Therefore, defining $\omega = 2\omega_1 + \omega_0$, we have that for n sufficiently large,

$$[\tilde{y}_n - \omega_1, \tilde{y}_n + \omega_1] \subset [y_n - \omega, y_n + \omega].$$

Hence

$$\int_{y_n-\omega}^{y_n+\omega} \left(|f_n|^2 + g_n^2\right)\ dx > \sigma.$$

In particular, we may take $\sigma = s + t - 1/k$, and thus we have shown that for every $k \in \mathbb{N}$ there exists $\omega_k \in \mathbb{R}$ such that for all sufficiently large n,

$$\int_{y_n-\omega_k}^{y_n+\omega_k} \left(|f_n|^2 + g_n^2\right)\ dx > s + t - \frac{1}{k}. \tag{3.28}$$

Let us now define $w_n(x) = f_n(x + y_n)$ and $z_n(x) = g_n(x + y_n)$. Then by (3.28), for every $k \in \mathbb{N}$, we have

$$\int_{-\omega_k}^{\omega_k} \left(|w_n|^2 + z_n^2\right)\ dx > s + t - \frac{1}{k}, \tag{3.29}$$

provided n is sufficiently large. Now by Lemma 3.9, $\{(w_n, z_n)\}$ is bounded in X, so there exists a subsequence, denoted again by $\{(w_n, z_n)\}$, which converges weakly in X to a limit $(f, g) \in X$. By Fatou's Lemma, $\|f\|^2 \leqq s$ and $\|g\|^2 \leqq t$. For each $k \in \mathbb{N}$, the inclusion of $H^1(-\omega_k, \omega_k)$ into $L^2(-\omega_k, \omega_k)$ is compact, so by passing to a subsequence, we may assume that $w_n \to f$ strongly in $L^2(-\omega_k, \omega_k)$. Furthermore, by using a diagonalization argument, we may assume that a single subsequence of $\{w_n\}$ has been chosen which has this property for every k. Now

$$\limsup_{n\to\infty} \int_{-\omega_k}^{\omega_k} z_n^2\ dx \le t,$$

so taking $n \to \infty$ in (3.29) gives

$$\int_{-\infty}^{\infty} |f|^2\ dx \ge \int_{-\omega_k}^{\omega_k} |f|^2\ dx = \lim_{n\to\infty} \int_{-\omega_k}^{\omega_k} |w_n|^2\ dx \ge s - \frac{1}{k}.$$

Since $\|f\|^2 \leq s$ and $k \in \mathbb{N}$ is arbitrary, this implies $\|f\|^2 = s$. Hence $w_n \to f$ strongly in L^2.

Next, observe that

$$\int_{-\infty}^{\infty} \left(z_n|w_n|^2 - g|f|^2\right)\ dx = \int_{-\infty}^{\infty} z_n(|w_n|^2 - |f|^2)\ dx + \int_{-\infty}^{\infty} (z_n - g)|f|^2\ dx, \tag{3.30}$$

and consider separately the behavior of the integrals on the right-hand side as $n \to \infty$. For the first integral, we have

$$\left|\int_{-\infty}^{\infty} z_n(|w_n|^2 - |f|^2)\ dx\right| \leq \|z_n\|\|w_n - f\| \left(\|w_n\|_1 + \|f\|_1\right),$$

and the right-hand side goes to zero since $\{(w_n, z_n)\}$ is bounded in X, f is in H^1, and $w_n \to f$ in L^2. The second integral on the right-hand side of (3.30) converges to zero because $f^2 \in L^2$ and z_n converges to g weakly in L^2. It follows then from (3.30) that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} z_n|w_n|^2\ dx = \int_{-\infty}^{\infty} g|f|^2\ dx. \tag{3.31}$$

Since, by Fatou's Lemma,

$$\int_{-\infty}^{\infty} |f'|^2\ dx \leq \liminf_{n\to\infty} \int_{-\infty}^{\infty} |w_n'|^2\ dx,$$

it follows that

$$I_1(s,t) = \lim_{n\to\infty} K(w_n, z_n) \geq \int_{-\infty}^{\infty} \left(|f'|^2 - g|f|^2\right)\ dx = K(f,g). \tag{3.32}$$

We now claim that $\|g\|^2 = t$. To see this, first observe that Lemma 3.3 and (3.32) imply that

$$\int_{-\infty}^{\infty} g|f|^2\ dx > 0. \tag{3.33}$$

In particular, (3.33) gives that $\|g\| \neq 0$. So $0 < \|g\|^2 \leqq t$, and we can define $\eta \geq 1$ by $\eta = \sqrt{t}/\|g\|$. Then $\|\eta g\|^2 = t$, so by (3.32)

$$\begin{aligned} I_1(s,t) \leqq K(f, \eta g) &= K(f,g) + (1-\eta) \int_{-\infty}^{\infty} g|f|^2\ dx \\ &\leqq I_1(s,t) + (1-\eta) \int_{-\infty}^{\infty} g|f|^2\ dx. \end{aligned}$$

But then (3.33) implies that $(1-\eta) \geqq 0$, so $\eta = 1$ and $\|g\|^2 = t$, as was claimed.

It follows that $\{z_n\}$ converges strongly to g, and that (f,g) is a minimizer for $I_1(s,t)$. To complete the proof of the Lemma, it remains only to observe that since equality holds in (3.32), then $\int_{-\infty}^{\infty} |w_n'|^2\ dx \to \int_{-\infty}^{\infty} |f'|^2\ dx$ as $n \to \infty$, and therefore w_n converges to f strongly in H^1. □

The variational problem in (3.5) can also be solved by the method of concentration compactness, and indeed this has already been done in several places in the literature (see, for example, Theorem 2.9 of [2]). However, in the results above, we have already done most of the work involved in the proof, so for the reader's convenience we sketch here the remainder of the proof. Assuming $t > 0$, one lets $\{g_n\}$ be any minimizing sequence for $I_2(t)$, and defines

$$\tilde{Q}_n(\omega) = \sup_{y\in\mathbb{R}} \int_{y-\omega}^{y+\omega} g_n^2(x)\ dx.$$

Again we may assume that $\tilde{Q}_n$ converges pointwise to a nondecreasing function $\tilde{Q}$ on $[0,\infty)$, and we define

$$\tilde{\alpha} = \lim_{\omega\to\infty} \tilde{Q}(\omega).$$

The same arguments as in the proofs of Lemmas 3.9 and 3.10 show that $\|g_n\|_1$ remains bounded, and that

$$I_2(\tilde{\alpha}) + I_2(t-\tilde{\alpha}) \le I_2(t).$$

But it then follows from (3.13) that $\tilde{\alpha} \notin (0,t)$, and as before we see from Lemma 3.8 that $\tilde{\alpha} \ne 0$. Hence $\tilde{\alpha} = t$, and using the same argument as in the proof of Theorem 3.12, we deduce the following result.

Theorem 3.13. *Let $t > 0$, and let $\{g_n\}$ be any minimizing sequence for $I_2(t)$. Then there is a subsequence $\{g_{n_k}\}$ and a sequence of real numbers $\{y_k\}$ such that*

$$g_{n_k}(\cdot + y_k)$$

converges strongly in H^1 to some $g \in H^1$. The limit g is a minimizer for $I_2(t)$; i.e, $\|g\|^2 = t$ and $J(g) = I_2(t)$.

As consequences of Theorems 3.12 and 3.13, we obtain explicit values for the constant $I_1(1,1)$ and $I_2(1)$.

Corollary 3.14. *For every $s, t \ge 0$,*

$$I_1(s,t) = A_1 s t^{2/3},$$

where $A_1 = I_1(1,1) = -(3/16)^{2/3}$.

Proof. We may assume $s,t > 0$. Let $(f,g) \in X$ be a minimizer for $I_1(s,t)$, whose existence is guaranteed by Theorem 3.12. Then f and g satisfy the Lagrange multiplier equations (2.11), in which λ and μ are the multipliers. Therefore, up to a phase factor and a translation, $f = f_1$ and $g = g_1$, where f_1 and g_1 are given in (2.12).

To determine the values of λ and μ, we substitute f_1 and g_1 into the constraint equations $\|f\|^2 = s$ and $\|g\|^2 = t$. Using the formula

$$\int_{-\infty}^{\infty} \operatorname{sech}^{2n}(x)dx = \frac{\Gamma(\frac{1}{2})\Gamma(n)}{\Gamma(\frac{2n+1}{2})}, \tag{3.34}$$

one finds that $\lambda = (3t/16)^{2/3}$, $\mu = s(12t)^{-1/3}$, and

$$K(f_1, g_1) = -4\lambda^{3/2}\mu = -s\left(\frac{3t}{16}\right)^{2/3}.$$

Since $I_1(s,t) = K(f_1, g_1)$, this completes the proof. □

Corollary 3.15. *For every $t \geq 0$,*

$$I_2(t) = A_2 t^{5/3},$$

where $A_2 = I_2(1) = -\frac{8}{5}(\frac{3}{8})^{5/3} q^{4/3}$.

Proof. We may assume $t > 0$. Let g be a minimizer for $I_2(t)$, whose existence is guaranteed by Theorem 3.13. Then g satisfies the Lagrange multiplier equation (2.13), in which κ is the multiplier. Therefore, up to translation, $g = g_2$, where g_2 is given in (2.14). From $\|g_2\|^2 = t$ and (3.34), we deduce that

$$\kappa = \left(\frac{3q^2 t}{8}\right)^{2/3}.$$

The statement of the Corollary then follows from the substitution of the formulas for $g_2(x)$ and κ into the expression

$$I_2(t) = J(g_2) = \int_{-\infty}^{\infty} \left((g_2')^2 - q g_2^3\right)\, dx$$

and using again (3.34). □

Lemma 3.16. *Suppose $s, t > 0$. Let (f_1, g_1) be a minimizer for $I_1(s,t)$, and let g_2 be a minimizer for $I_2(t)$. Then*

$$M(g_2) = A_3 t^{2/3},$$

where

$$A_3 = \frac{-2 \cdot 3^{2/3} q^{1/3}}{q + 8 + \sqrt{q^2 + 16q}}. \tag{3.35}$$

Proof. The proof of (3.35) depends on being able to find explicitly the minimizing function f for $K(f, g_2)$ on the constraint set $\{\|f\| = 1\}$. The Lagrange multiplier equation for this variational problem is

$$-f'' - f g_2 = \lambda f, \tag{3.36}$$

so we see that the minimizer f is an eigenfunction for the Schrödinger operator $\mathcal{L} = -\frac{d^2}{dx^2} - g_2$ with potential g_2, and the Lagrange multiplier λ is the eigenvalue corresponding to f. Further, multiplying (3.36) by f and integrating over $\mathbb{R}$, we see that the constant C being sought is actually the same as the least or ground-state eigenvalue λ, so that f is a ground-state eigenfunction.

Now, $g_2(x) = a\,\text{sech}^2(bx)$, where a and b are constants; and for such potentials, with arbitrary positive values of a and b, the complete solution of the spectral problem for $\mathcal{L}$ is well-known (see, for example, [31, p. 768 ff.]). It turns out that the ground-state eigenfunction is a constant multiple of $\operatorname{sech}^p(bx)$, where

$$p = \sqrt{\left(\frac{a}{b^2}\right) + \frac{1}{4}} - \frac{1}{2}, \tag{3.37}$$

and the corresponding eigenvalue is

$$\lambda = -b^2 p^2. \tag{3.38}$$

In the proof of Corollary 3.15 we saw that the particular values of a and b corresponding to our potential g_2 are $a = \kappa/q$ and $b = \frac{\sqrt{\kappa}}{2}$, where $\kappa = (\frac{3q^2t}{8})^{2/3}$. Using these values to compute p and λ from (3.37) and (3.38), we obtain the asserted value for $A_3 = \lambda/t^{2/3}$. □

Corollary 3.17. *For $s, t \geq 0$ we have*

$$A_1 st^{2/3} + A_2 t^{5/3} \leq I(s,t), \tag{3.39}$$

and

$$I(s,t) \leq A_3 st^{2/3} + A_2 t^{5/3}. \tag{3.40}$$

Proof. From (3.3), we have

$$I_1(s,t) + I_2(t) \leq I(s,t),$$

which, in view of Corollaries 3.14 and 3.15, yields (3.39). To prove (3.40), let g_2 be as in Lemma 3.16. Then Lemma 3.16 and (3.3) give

$$I(s,t) \leq sM(g_2) + J(g_2) \leq A_3 st^{2/3} + A_2 t^{5/3}.$$

□

Remark 3.18. The case when $q = 2$ is special, because then the function g_1 defined in Corollary 3.14 coincides with the function g_2 defined in Corollary 3.15. It follows that in this case $A_1 = A_3$, and hence

$$I(s,t) = A_1 st^{2/3} + A_2 t^{5/3}.$$

Moreover, the pair (f_1, g_1) defined in Corollary 3.14 is an explicit minimizer for the problem (3.1). In fact, it follows from the uniqueness of the solutions of (3.4) and (3.5) that (f_1, g_1) is the unique minimizer for (3.1) (up to a translation in x and a multiplication of f_1 by a constant of absolute value 1). This is the case analyzed by Chen in [15].

Our next goal is to investigate the subadditivity of $I(s,t)$. The preceding corollary and remark suggest the strategy of comparing $I(s,t)$ with a function of the type $At^{5/3} + Bst^{2/3}$, which, as was seen in the proof of Lemma 3.7, is subadditive when A and B are negative constants. The next few lemmas are devoted to showing that $I(s,t)$ is close enough to a function of this type to inherit the property of subadditivity.

Lemma 3.19. *Suppose $s,t \geq 0$. Then we can find a sequence $\{g_n^{s,t}\}$ in H^1 such that $\lim_{n\to\infty} M(g_n^{s,t}) = M_0(s,t)$ and $\lim_{n\to\infty} J(g_n^{s,t}) = J_0(s,t)$ exist and satisfy*

(i) $sM_0(s,t) + J_0(s,t) = I(s,t)$,

(ii) $A_1 st^{2/3} \leq sM_0(s,t) \leq A_3 st^{2/3}$, and

(iii) $A_2 t^{5/3} \leq J_0(s,t) \leq A_2 t^{5/3} + (A_3 - A_1) st^{2/3}$.

Proof. Let $\{g_n^{s,t}\}$ be any minimizing sequence for $I(s,t)$ in the strict sense; i.e, a sequence in H^1 such that $\|g_n^{s,t}\|^2 = t$ and

$$\lim_{n\to\infty} \left(sM(g_n^{s,t}) + J(g_n^{s,t}) \right) = I(s,t). \tag{3.41}$$

Since $\{M(g_n^{s,t})\}$ and $\{J(g_n^{s,t})\}$ are bounded sequences of real numbers, by passing to a subsequence we may assume that the limits $M_0(s,t)$ and $J_0(s,t)$ exist as defined above. Then (i) follows immediately from (3.41).

Next, observe that Corollaries 3.14 and 3.15 imply that

$$A_1 st^{2/3} \leq sM_0(s,t) \tag{3.42}$$

and

$$A_2 t^{5/3} \leq J_0(s,t). \tag{3.43}$$

From (i), (3.40), and (3.42) we get

$$A_1 st^{2/3} + J_0(s,t) \leq A_3 st^{2/3} + A_2 t^{5/3},$$

which implies the upper bound in (iii). From (i), (3.40), and (3.43), we get

$$sM_0(s,t) + A_2 t^{5/3} \leq A_3 st^{2/3} + A_2 t^{5/3},$$

which implies the upper bound in (ii). □

Remark 3.20. As defined above in Lemma 3.19, the quantities $M_0(s,t)$ and $J_0(s,t)$ could depend on the choice of the minimizing sequence $\{g_n^{s,t}\}$, as well as on s and t. This ambiguity of notation will not affect the validity of the statements which follow.

Lemma 3.21. *Suppose $s_1, s_2, t_1, t_2 \geq 0$ with $s_2 t_1 > s_1 t_2$. Then*

$$t_2^{5/3} J_0(s_1,t_1) \leq t_1^{5/3} J_0(s_2,t_2) \tag{3.44}$$

and

$$t_2^{2/3} M_0(s_1,t_1) \geq t_1^{2/3} M_0(s_2,t_2). \tag{3.45}$$

Proof. The inequalities are obvious when $t_2 = 0$, so we may assume that $t_2 > 0$, and hence also $t_1 > 0$. Let $\sigma = t_1/t_2$, and for any $g \in H^1$ define g^* as in Lemma 3.4. Then for all $n \in \mathbb{N}$, $\|(g_n^{s_2,t_2})^*\|^2 = t_1$, so by (3.3), Lemma 3.4, and Lemma 3.19(i), we have

$$\begin{aligned} s_1 M_0(s_1,t_1) + J_0(s_1,t_1) = I(s_1,t_1) &= \inf\{s_1 M(g) + J(g) : \|g\|^2 = t_1\} \\ &\leq s_1 M((g_n^{s_2,t_2})^*) + J((g_n^{s_2,t_2})^*) \\ &= s_1 \sigma^{2/3} M(g_n^{s_2,t_2}) + \sigma^{5/3} J(g_n^{s_2,t_2}). \end{aligned}$$

Taking $n \to \infty$ then gives

$$s_1 M_0(s_1,t_1) + J_0(s_1,t_1) \le s_1 \sigma^{2/3} M_0(s_2,t_2) + \sigma^{5/3} J_0(s_2,t_2). \tag{3.46}$$

Similarly, we obtain

$$s_2 M_0(s_2,t_2) + J_0(s_2,t_2) \le s_2 \sigma^{-2/3} M_0(s_1,t_1) + \sigma^{-5/3} J_0(s_1,t_1). \tag{3.47}$$

Multiplying (3.46) by s_2 and (3.47) by $s_1\sigma^{2/3}$, and adding the results, we obtain

$$\sigma^{-5/3} J_0(s_1,t_1)(s_2\sigma - s_1) \le J_0(s_2,t_2)(s_2\sigma - s_1).$$

Since $s_2\sigma - s_1 > 0$, this implies (3.44). Similarly, multiplying (3.47) by $\sigma^{5/3}$, adding to (3.47), and rearranging, we obtain

$$\sigma^{2/3} M_0(s_2,t_2)(s_2\sigma - s_1) \le M_0(s_1,t_1)(s_2\sigma - s_1),$$

which implies (3.45). □

Lemma 3.22. *Suppose $s_1, s_2, t_1, t_2 > 0$. Let $\eta = t_1/t_2$.*

(i) If

$$\eta > |A_1/A_3|^{3/2} - 1, \tag{3.48}$$

then

$$(1 + 1/\eta)^{2/3} M_0(s_1,t_1) < M_0(s_2,t_2). \tag{3.49}$$

(ii) Let $\alpha(\eta) = ((1+\eta)^{2/3} - 1)\eta^{-5/3}$. If

$$\alpha(\eta) > |A_1/A_3| - 1, \tag{3.50}$$

then

$$s_2 \left[(1+\eta)^{2/3} - 1\right] M_0(s_2,t_2) < J_0(s_1,t_1) + J_0(s_2,t_2) - (1+\eta)^{5/3} J_0(s_2,t_2). \tag{3.51}$$

Proof. Since $s_1 > 0$, we can use Lemma 3.19(ii) to write

$$(1 + 1/\eta)^{2/3} M_0(s_1,t_1) \le (1 + 1/\eta)^{2/3} A_3 t_1^{2/3} = (t_1 + t_2)^{2/3} A_3$$

and

$$A_1 t_2^{2/3} \le M(s_2,t_2).$$

Combining these inequalities with (3.48), we obtain (3.49). This proves (i).

To prove (ii), use Lemma 3.19(ii) to write

$$s_2 \left[(1+\eta)^{2/3} - 1\right] M_0(s_2,t_2) \le s_2 \left[(1+\eta)^{2/3} - 1\right] A_3 t_2^{2/3},$$

and use Lemma 3.19(iii) to write

$$\begin{aligned} J_0(s_1,t_1) + J_0(s_2,t_2) - (1+\eta)^{5/3} J_0(s_2,t_2) &\ge J_0(s_1,t_1) - \eta^{5/3} J_0(s_2,t_2) \\ &\ge A_2 t_1^{5/3} - \eta^{5/3}\left(A_2 t_2^{5/3} + s_2(A_3 - A_1) t_2^{2/3}\right) \\ &= -s_2 \eta^{5/3} |A_3 - A_1| t_2^{2/3}. \end{aligned}$$

Also, (3.50) implies that

$$A_3((1-\eta)^{2/3} - 1) < |A_3 - A_1| \eta^{5/3}.$$

Combining these inequalities gives (3.51). □

Now define $\eta_1(q) = (|A_1/A_3|^{3/2} - 1)$, and define $\eta_2(q)$ to be the value of η for which the right- and left-hand sides of (3.50) are equal. (When the right-hand side is zero, we can take $\eta_2(q) = \infty$.) If $\eta_2(q) > \eta_1(q)$, then any positive real number η satisfies at least one of the inequalities (3.48) or (3.50). Analysis of the functions η_1 and η_2 shows that there does exist a nonempty interval (q_1, q_2) of values of q for which the inequality $\eta_2(q) > \eta_1(q)$ is valid. In fact, when $q = 2$, one has $A_1 = A_3$ (see Remark 3.18), so $\eta_1(2) = 0$, while $\eta_2(2) = \infty$. Therefore the interval (q_1, q_2) contains at least a neighborhood of $q = 2$. On the other hand, as $q \to 0$ or $q \to \infty$, one has $\eta_1(q) \to \infty$ and $\eta_2(q) \to 0$, so the interval (q_1, q_2) is bounded above and bounded away from zero.

We can now prove that $I(s,t)$ is subadditive, at least when $q \in (q_1, q_2)$.

Theorem 3.23. *Suppose $q \in (q_1, q_2)$. Let $s_1, s_2, t_1, t_2 \geq 0$, and suppose that $s_1 + s_2 > 0$, $t_1 + t_2 > 0$, $s_1 + t_1 > 0$, and $s_2 + t_2 > 0$. Then*

$$I(s_1 + s_2, t_1 + t_2) < I(s_1, t_1) + I(s_2, t_2). \tag{3.52}$$

Proof. We may assume without loss of generality that $s_2 t_1 \geq s_1 t_2$. If $s_2 t_1 = s_1 t_2$, then our assumptions imply that s_1, s_2, t_1, and t_2 must all be positive, and since $(t_1 + t_2)/t_2 = (s_1 + s_2)/s_2$, we can write

$$\begin{aligned} I(s_1 + s_2, t_1 + t_2) &= \left(\frac{t_1 + t_2}{t_2}\right)^{5/3} I(s_2, t_2) = \left(1 + \frac{t_1}{t_2}\right)^{5/3} I(s_2, t_2) \\ &< \left[1 + \left(\frac{t_1}{t_2}\right)^{5/3}\right] I(s_2, t_2) = I(s_2, t_2) + I(s_1, t_1). \end{aligned}$$

Here we have twice used Lemma 3.6, and have also used the fact that $I(s_2, t_2) < 0$, which is a consequence of Lemma 3.17.

We may therefore assume that $s_2 t_1 > s_1 t_2$, and in particular that $s_2 > 0$ and $t_1 > 0$. For now, we assume also that $t_2 > 0$, and we define $\eta = t_1/t_2$. From our hypothesis on q we know that η satisfies either (3.48) or (3.50); we consider the two cases separately.

In case (3.48) holds, define $\sigma = 1 + 1/\eta$ and $h_n(x) = \sigma^{2/3} g_n^{s_1,t_1}(\sigma^{1/3} x)$. By passing to a subsequence if necessary, we may assume that $J(h_n)$ and $M(h_n)$ converge as $n \to \infty$. Then using Lemma 3.4 and (3.44), we get

$$\begin{aligned} \lim_{n\to\infty} J(h_n) &= \sigma^{5/3} J_0(s_1, t_1) \\ &\leq J_0(s_1, t_1) + \left(\frac{t_2}{t_1}\right)^{5/3} J_0(s_1, t_1) \\ &\leq J_0(s_1, t_1) + J_0(s_2, t_2). \end{aligned} \tag{3.53}$$

Next, suppose that $s_1 > 0$. Then from Lemma 3.4 and (3.49) we have

$$\begin{aligned} (s_1 + s_2) \lim_{n\to\infty} M(h_n) &= (s_1 + s_2)\sigma^{2/3} M_0(s_1, t_1) \\ &\leq s_1 M_0(s_1, t_1) + s_2 \sigma^{2/3} M_0(s_1, t_1) \\ &< s_1 M_0(s_1, t_1) + s_2 M_0(s_2, t_2). \end{aligned} \tag{3.54}$$

Now, since $\|h_n\|^2 = t_1 + t_2$, we get from (3.53) and (3.54) that

$$\begin{aligned} I(s_1+s_2, t_1+t_2) &\leqq (s_1+s_2) \lim_{n\to\infty} M(h_n) + \lim_{n\to\infty} J(h_n) \\ &< s_1 M_0(s_1,t_1) + s_2 M_0(s_2,t_2) + J_0(s_1,t_1) + J_0(s_2,t_2) \\ &= I(s_1,t_1) + I(s_2,t_2), \end{aligned}$$

as desired.

If, on the other hand, $s_1 = 0$, then we cannot use the above argument, since (3.54) does not hold. Instead we use Corollary 3.17 and (3.48) to write

$$\begin{aligned} I(0+s_2, t_1+t_2) &\leq A_3 s_2 (t_1+t_2)^{2/3} + A_2 (t_1+t_2)^{5/3} \\ &\leq A_3 s_2 (1+1/\eta)^{2/3} t_2^{2/3} + A_2 t_1^{5/3} + A_2 t_2^{5/3} \\ &< A_1 s_2 t_2^{2/3} + A_2 t_2^{5/3} + A_2 t_1^{5/3} \\ &\leq I(s_2,t_2) + I_2(t_1) = I(s_2,t_2) + I(0,t_1), \end{aligned}$$

which again gives (3.52).

In case (3.50) holds, we define $j_n(x) = \sigma^{2/3} g_n^{s_2,t_2}(\sigma^{1/3}x)$, where $\sigma = 1+\eta$. Again we may assume that $M(j_n)$ and $J(j_n)$ converge, and since $\|j_n\|^2 = t_1+t_2$, we have

$$I(s_1+s_2, t_1+t_2) \leq (s_1+s_2) \lim_{n\to\infty} M(j_n) + \lim_{n\to\infty} J(j_n).$$

It follows from Lemma 3.4 that

$$I(s_1+s_2, t_1+t_2) \leq (s_1+s_2)\sigma^{2/3} M_0(s_2,t_2) + \sigma^{5/3} J_0(s_2,t_2).$$

Now from (3.45), we have

$$\sigma^{2/3} M_0(s_2,t_2) < \eta^{2/3} M_0(s_2,t_2) \leq M_0(s_1,t_1),$$

so

$$I(s_1+s_2, t_1+t_2) < s_1 M_0(s_1,t_1) + s_2 \sigma^{2/3} M_0(s_2,t_2) + \sigma^{5/3} J_0(s_2,t_2).$$

Also, from (3.51) we have

$$s_2 \sigma^{2/3} M_0(s_2,t_2) + \sigma^{5/3} J_0(s_2,t_2) < s_2 M_0(s_2,t_2) + J_0(s_1,t_1) + J_0(s_2,t_2).$$

Combining the last two inequalities, we get (3.52).

Finally, it remains to consider the case when $t_2 = 0$, which implies $I(s_2,t_2) = 0$ by Corollary 3.17. If $s_1 > 0$, then $M_0(s_1,t_1) < 0$ by Lemma 3.19(ii), so letting $h_n = g_n^{s_1,t_1}$, we have

$$\begin{aligned} I(s_1+s_2, t_1) &\leq (s_1+s_2) \lim_{n\to\infty} M(h_n) + \lim_{n\to\infty} J(h_n) \\ &= (s_1+s_2) M_0(s_1,t_1) + J_0(s_1,t_1) \\ &< s_1 M_0(s_1,t_1) + J_0(s_1,t_1) = I(s_1,t_1) = I(s_1,t_1) + I(s_2,t_2). \end{aligned}$$

If, on the other hand, $s_1 = 0$, then we use Corollary 3.17 to write

$$I(s_2,t_1) \leq A_3 s_2 t_1^{2/3} + A_2 t_1^{5/3} < A_2 t_1^{5/3} = I_2(t_1) = I(0,t_1) = I(0,t_1) + I(s_2,0),$$

and we are done. □

Lemma 3.24. *Suppose $s,t>0$. If $\{(f_n,g_n)\}$ is a minimizing sequence for $I(s,t)$, then $\{(f_n,g_n)\}$ is bounded in Y.*

Proof. For a minimizing sequence, $\|f_n\|$ and $\|g_n\|$ stay bounded, so as in the proof of Lemma 3.9, we have that

$$\left|\int_{-\infty}^{\infty} g_n|f_n|^2\ dx\right| \le C\|f_n\|_1^{1/2},$$

where C is independent of n. Also, Sobolev embedding and interpolation theorems give

$$\left|\int_{-\infty}^{\infty} g_n^3\ dx\right| \le |g_n|_3^3 \le C\|g_n\|_{1/6}^3 \le C\|g_n\|_1^{1/2}\|g_n\|^{5/2} \le C\|g_n\|_1^{1/2}.$$

Hence

$$\begin{aligned}\|(f_n,g_n)\|_Y^2 &= \|f_n\|_1^2 + \|g_n\|_1^2\\ &= E(f_n,g_n) + \int_{-\infty}^{\infty} g_n|f_n|^2\ dx + \int_{-\infty}^{\infty} g_n^3\ dx + \|f_n\|^2 + \|g_n\|^2\\ &\le C(1+\|f_n\|_1^{1/2} + \|g_n\|_1^{1/2}) \le C(1+\|(f_n,g_n)\|_Y^{1/2}),\end{aligned}$$

from which the desired conclusion follows. □

Now we establish the relative compactness, up to translations, of minimizing sequences for $I(s,t)$. The idea again is to use the method of concentration compactness. Let $\{(f_n,g_n)\}$ be a minimizing sequence for $I(s,t)$, and let $P_n(\omega)$ be the sequence of nondecreasing functions defined for $\omega>0$ by

$$P_n(\omega) = \sup_{y\in\mathbb{R}} \int_{y-\omega}^{y+\omega} \left(|f_n|^2(x) + g_n^2(x)\right)\ dx.$$

Then $\{P_n\}$ has a pointwise convergent subsequence on $[0,\infty)$, which we denote again by $\{P_n\}$. Let P be the nondecreasing function to which P_n converges, and define

$$\alpha_0 = \lim_{\omega\to\infty} P(\omega). \tag{3.55}$$

Then, as was true for α in (3.14), we have $0 \leqq \alpha_0 \leqq s+t$.

Lemma 3.25. *Suppose $s,t>0$, and let $\{(f_n,g_n)\}$ be any minimizing sequence for $I(s,t)$. Let α_0 be as defined in (3.55). Then there exist numbers $s_1\in[0,s]$ and $t_1\in[0,t]$ such that*

$$s_1+t_1=\alpha_0 \tag{3.56}$$

and

$$I(s_1,t_1) + I(s-s_1,t-t_1) \le I(s,t). \tag{3.57}$$

Proof. As in the proof of Lemma 3.10, we can define sequences $\{(\varphi_n,h_n)\}$ and $\{(l_n,j_n)\}$ in Y such that $\|\varphi_n\|^2\to s_1$, $\|h_n\|^2\to t_1$, $\|l_n\|^2\to s-s_1$, $\|j_n\|^2\to t-t_1$, $E(\varphi_n,h_n)\to E_1$, and $E(l_n,j_n)\to E_2$, where $s_1\in[0,s]$ and $t_1\in[0,t]$ satisfy (3.56) and

$$E_1+E_2 \le I(s,t).$$

The only change that has to be made is that in place of the estimates (3.19), (3.20), and (3.21) for the functional K, we must put similarly obtained estimates for the functional E.

To complete the proof of the lemma, then, it only remains to show that $E_1 \geq I(s_1, t_1)$ and $E_2 \geq I(s-s_1, t-t_1)$. We need only prove the first of these inequalities, since the proof of the second is similar. As in the proof of (3.24) we consider separately the three cases when $s_1 > 0$ and $t_1 > 0$, when $s_1 = 0$ and $t_1 > 0$, and when $t_1 = 0$. When $s_1 > 0$ and $t_1 > 0$, we use the same argument as was used in this case for (3.24). When $s_1 = 0$, then $\|\varphi_n\| \to 0$, so (3.26) is established by the same proof as before. Then we have, as in (3.27),

$$E_1 = \lim_{n\to\infty} E(\varphi_n, h_n) = \lim_{n\to\infty} \left(K(\varphi_n, h_n) + J(h_n)\right) \geq \liminf_{n\to\infty} J(h_n).$$

Also, since $\|h_n\| > 0$ for n large, we can put $\theta_n = \sqrt{t_1}/\|h_n\|$, and we have

$$I(0, t_1) = J(t_1) \leq J(\theta_n h_n) \leq \liminf_{n\to\infty} J(h_n),$$

since $\theta_n \to 1$. Therefore $E_1 \geq I(0, t_1)$. Finally, if $t_1 = 0$ then $\|h_n\| \to 0$, so (3.26) still holds, and moreover

$$\left| \int_{-\infty}^{\infty} h_n^3 \, dx \right| \leq \|h_n\|_1 \|h_n\|^2 \to 0.$$

Therefore

$$E_1 = \lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|\varphi_n'|^2 - h_n|\varphi_n|^2 + (h_n')^2 - h_n^3\right) \, dx \geq 0 = I(s_1, 0).$$

□

Theorem 3.26. *Suppose $q \in (q_1, q_2)$, and let $s, t > 0$. Then every minimizing sequence $\{(f_n, g_n)\}$ for $I(s,t)$ is relatively compact in Y up to translations; i.e., there is a subsequence $\{(f_{n_k}, g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that*

$$(f_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k))$$

converges strongly in Y to some (f, g), which is a minimizer for $I(s,t)$.

Proof. If $\alpha_0 = 0$, then as in the proof of Lemma 3.11, we get $|f_n|_4 \to 0$ and $|g_n|_3 \to 0$ as $n \to \infty$, whence

$$I(s,t) = \lim_{n\to\infty} E(f_n, g_n) \geq \liminf_{n\to\infty} \int \left(|f_n'|^2 + (g_n')^2\right) \, dx \geq 0,$$

contradicting Corollary 3.17. Hence $\alpha_0 > 0$. On the other hand, if $\alpha_0 \in (0, s+t)$ then it follows from Theorem 3.23 that

$$I(s,t) < I(s_1, t_1) + I(s - s_1, t - t_1),$$

which contradicts (3.57). Therefore we must have $\alpha_0 = s + t$.

It now follows, as in the proof of Theorem 3.12, that we can find real numbers $\{y_n\}$ such that if $w_n(x) = f_n(x+y_n)$ and $z_n(x) = g_n(x+y_n)$, then for every $k \in \mathbb{N}$, there exists $\omega_k \in \mathbb{R}$ such that

$$\int_{-\omega_k}^{\omega_k} \left(|w_n|^2 + z_n^2\right)\ dx > s + t - \frac{1}{k}, \tag{3.58}$$

provided n is sufficiently large (cf. (3.29)). Since the sequence $\{(w_n, z_n)\}$ is bounded in Y, there exists a subsequence, denoted again by $\{(w_n, z_n)\}$, which converges weakly in Y to a limit (f, g). Then Fatou's Lemma implies that

$$\|f\|^2 + \|g\|^2 \leq \liminf_{n\to\infty} \int_{-\infty}^{\infty} \left(|w_n|^2 + z_n^2\right)\ dx = s + t.$$

Moreover, for fixed k, (w_n, z_n) converges weakly in $H^1(-\omega_k, \omega_k) \times H^1(-\omega_k, \omega_k)$ to (f, g), and therefore has a subsequence, denoted again by $\{(w_n, z_n)\}$, which converges strongly to (f, g) in $L^2(-\omega_k, \omega_k) \times L^2(-\omega_k, \omega_k)$. By a diagonalization argument, we may assume that the subsequence has this property for every k simultaneously. It follows then from (3.58) that

$$\int_{-\infty}^{\infty} \left(|f|^2 + g^2\right)\ dx \geqq \int_{-\omega_k}^{\omega_k} \left(|f|^2 + g^2\right)\ dx \geqq s + t - \frac{1}{k}.$$

Since k was arbitrary, we get $\int_{-\infty}^{\infty} \left(|f|^2 + g^2\right)\ dx = s+t$, which implies that (w_n, z_n) converges strongly to (f, g) in $L^2 \times L^2$.

Now we have that $\int_{-\infty}^{\infty} z_n |w_n|^2\ dx \to \int_{-\infty}^{\infty} g|f|^2\ dx$ as $n \to \infty$, by the same argument used to establish (3.31), or by an even simpler argument which uses the strong convergence of z_n to g in L^2. Moreover,

$$|z_n - g|_3 \leq C\|z_n - g\|_1^{1/6} \|z_n - g\|^{5/6} \leq C\|z_n - g\|^{5/6},$$

so $\int_{-\infty}^{\infty} z_n^3\ dx \to \int_{-\infty}^{\infty} g^3\ dx$. Therefore, by another application of Fatou's Lemma, we get

$$I(s, t) = \lim_{n\to\infty} E(w_n, z_n) \geqq E(f, g), \tag{3.59}$$

whence $E(f, g) = I(s, t)$. Thus (f, g) is a minimizer for the variational problem (3.1). Finally, since equality holds in (3.59), then

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} \left(|w_n'|^2 + (z_n')\right)\ dx = \int_{-\infty}^{\infty} \left(|f'|^2 + (g')^2\right)\ dx,$$

so (w_n, z_n) converges strongly to (f, g) in Y. □

For each $s > 0$ and $t > 0$, define $G_{s,t}$ to be the set of solutions to the variational problem (3.1); that is,

$$G_{s,t} = \{(f, g) \in Y : E(f, g) = I(s, t),\ \|f\|^2 = s, \text{ and } \|g\|^2 = t\}.$$

As a consequence of Theorem 3.26, we have that $G_{s,t}$ is non-empty for all $s, t > 0$, provided $q \in (q_1, q_2)$. As will be seen below in Section 5, this translates into an existence result for ground-state solutions of (1.2).

We next present a somewhat weaker version of Theorem 3.26 that is valid for all $q > 0$. For $\gamma > 0$, define $Q_\gamma : Y \to \mathbb{R}$ by

$$Q_\gamma(f,g) = \int_{-\infty}^{\infty} \left(|f|^2 + \gamma g^2\right)\, dx,$$

and for each $\beta > 0$, define

$$R(\beta,\gamma) = \inf\left\{E(f,g) : (f,g) \in Y \text{ and } Q_\gamma(f,g) = \beta\right\}. \tag{3.60}$$

Theorem 3.27. *Suppose $q > 0$ and let $\beta, \gamma > 0$. Then every minimizing sequence $\{(f_n, g_n)\}$ for $R(\beta,\gamma)$ is relatively compact in Y up to translations; i.e., there is a subsequence $\{(f_{n_k}, g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that*

$$(f_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k))$$

converges strongly in Y to some (f,g), which is a minimizer for $R(\beta,\gamma)$.

Proof. This theorem follows from the proof of Theorem 2.1 of [3]. First note that, if we decompose f into its real and imaginary parts as $f = \eta + i\theta$, and define $z : \mathbb{R} \to \mathbb{R}^3$ by $z = (\eta, \theta, g)$, then in the notation of [3] we have

$$E(f,g) = \int_{-\infty}^{\infty} \left(\frac{1}{2}\, \langle z, Lz\rangle - N(z)\right)\, dx$$

and

$$Q_\gamma(f,g) = \int_{-\infty}^{\infty} \frac{1}{2}\, \langle z, Dz\rangle\, dx,$$

where $Lz = -2z_{xx}$, $N(z) = g(\eta^2 + \theta^2 + qg^2)$, and $Dz = 2(\eta, \theta, \gamma g)$. Also, in the notation of [3], we have $\sigma_0 = 0$. Therefore the variational problem (3.60) is the same as the problem which defines I_β in [3], and $R(\beta,\gamma) = I_\beta$. It is easily verified that L, N, and D satisfy the conditions in Section 2 of [3]. To check that $I_\beta < 0$ for all $\beta > 0$, we can either use the identity

$$R(\beta,\gamma) = \inf\left\{I(s,t) : s > 0,\ t > 0, \text{ and } s + \gamma t = \beta\right\} \tag{3.61}$$

in conjunction with (3.17), or use Theorem 2.2 of [3]. Therefore all the hypotheses of Theorem 2.1 of [3] are verified, and we conclude from the proof of that Theorem that every minimizing sequence for $R(\beta,\gamma)$ is relatively compact in Y up to translations. □

To compare the results in Theorems 3.26 and 3.27, let us consider the sets

$$Q_{\beta,\gamma} = \left\{(f,g) \in Y : E(f,g) = R(\beta,\gamma) \text{ and } \int_{-\infty}^{\infty} \left(|f|^2 + \gamma g^2\right)\, dx = \beta\right\}$$

of solutions to problem (3.60). A consequence of Theorem 3.27 is that $Q_{\beta,\gamma}$ is non-empty for all $\beta, \gamma > 0$, regardless of the value of $q > 0$. In particular, from (3.61) it follows that if $Q_{\beta,\gamma}$ is non-empty then so is $G_{s,t}$, for some values of s and t satisfying $s + \gamma t = \beta$. One drawback, however, is that we do not know whether the sets $Q_{\beta,\gamma}$ constitute a true two-parameter family of disjoint sets. In particular, it is not clear whether every pair $s, t > 0$ corresponds to a pair β, γ such that $Q_{\beta,\gamma} \subseteq G_{s,t}$. A related drawback to Theorem 3.27 is that it does not lend itself as easily as does Theorem 3.26 to a result on ground-state solutions of (1.2). See Remark 4.6 below.

4. The full variational problem

We consider the problem of finding, for any $s > 0$ and $t \in \mathbb{R}$,

$$W(s,t) = \inf \{E(h,g) : (h,g) \in Y,\ H(h) = s, \text{ and } G(h,g) = t\}. \tag{4.1}$$

Following our usual convention, we define a minimizing sequence for $W(s,t)$ to be a sequence (h_n, g_n) in Y such that $H(h_n) \to s$, $G(h_n, g_n) \to t$, and $E(h_n, g_n) \to W(s,t)$ as $n \to \infty$.

Lemma 4.1. *Suppose $s > 0$ and $t \in \mathbb{R}$. If $\{(h_n, g_n)\}$ is a minimizing sequence for $W(s,t)$, then $\{(h_n, g_n)\}$ is bounded in Y.*

Proof. For a minimizing sequence, $\|h_n\| = \sqrt{H(h_n)}$ stays bounded, and since

$$\|g_n\|^2 = G(h_n, g_n) + 2 \text{ Im} \int_{-\infty}^{\infty} h_n (\overline{h_n})_x \, dx,$$

it follows that $\|g_n\|^2 \leq C(1 + \|h_n\|_1)$, where C is independent of n. Arguing as in the proofs of Lemmas 3.9 and 3.24, we deduce that

$$\left| \int_{-\infty}^{\infty} g_n |h_n|^2 \, dx \right| \leq C \|h_n\|_1^{1/2} \|g_n\| \leq C(1 + \|h_n\|_1)$$

and

$$\left| \int_{-\infty}^{\infty} g_n^3 \, dx \right| \leq C \|g_n\|_1^{1/2} \|g_n\|^{5/2} \leq C \|g_n\|_1^{1/2} \left(1 + \|h_n\|_1^{5/4}\right).$$

Hence, as in the proof of Lemma 3.24, we get

$$\|(h_n, g_n)\|_Y^2 \leq C \left(1 + \|h_n\|_1 + \|g_n\|_1^{1/2} (1 + \|h_n\|_1^{5/4})\right) \leq C(1 + \|(h_n, g_n)\|_Y^{7/4}),$$

which is sufficient to bound $\|(h_n, g_n)\|_Y$. □

Lemma 4.2. *Suppose $k, \theta \in \mathbb{R}$ and $h \in H^1_{\mathbb{C}}$. If $f(x) = e^{i(kx+\theta)} h(x)$, then*

$$E(f,g) = E(h,g) + k^2 H(h) - 2k \text{ Im} \int_{-\infty}^{\infty} h \overline{h}_x \, dx$$

and

$$G(f,g) = G(h,g) + 2kH(h).$$

We omit the proof, which is elementary.

Now we can establish a relation between problems (4.1) and (3.1).

Lemma 4.3. *Suppose $s > 0$ and $t \in \mathbb{R}$, and define $b = b(a) = (a - t)/(2s)$ for $a \geq 0$. Then*

$$W(s,t) = \inf_{a \geq 0} \left\{ I(s,a) + b(a)^2 s \right\} \tag{4.2}$$

and

$$W(s,t) < I(s,0) + b(0)^2 s. \tag{4.3}$$

Proof. First, suppose $a \geq 0$, and let $(h, g) \in Y$ be given such that $\|h\|^2 = s$ and $\|g\|^2 = a$. Let $b = b(a)$ and $c = \operatorname{Im} \int_{-\infty}^{\infty} h\overline{h}_x \, dx$, and put $f(x) = e^{ikx}h(x)$ with $k = (c/s) - b$. Then from Lemma 4.2 we deduce that

$$E(f, g) = E(h, g) - \frac{c^2}{s} + b^2 s \leq E(h, g) + b^2 s$$

and

$$G(f, g) = \|g\|^2 - 2bs = t.$$

Since $H(f) = s$, we conclude that

$$W(s, t) \leq E(f, g) \leq E(h, g) + b^2 s.$$

Taking the infimum over h and g gives

$$W(s, t) \leq I(s, a) + b^2 s,$$

and now taking the infimum over a gives

$$W(s, t) \leq \inf_{a \geq 0} \left\{ I(s, a) + b(a)^2 s \right\}. \tag{4.4}$$

Next, suppose $(h, g) \in Y$ is given such that $H(h) = s$ and $G(h, g) = t$. Define $a = t + 2 \operatorname{Im} \int_{-\infty}^{\infty} h\overline{h}_x \, dx$ and $f(x) = e^{ibx}h(x)$, where $b = b(a)$. Then by Lemma 4.2,

$$E(f, g) = E(h, g) + b^2 s - b(a - t) = E(h, g) - b^2 s,$$

and since $\|f\|^2 = s$ and $\|g\|^2 = a$, we have $a \geq 0$ and $I(s, a) \leq E(f, g)$. Hence

$$E(h, g) \geq I(s, a) + b^2 s \geq \inf_{a \geq 0} \left\{ I(s, a) + b(a)^2 s \right\},$$

and taking the infimum over h and g gives

$$W(s, t) \geq \inf_{a \geq 0} \left\{ I(s, a) + b(a)^2 s \right\}. \tag{4.5}$$

Combining (4.4) and (4.5) gives (4.2).

To prove (4.3), we see from (4.4) that it suffices to show there exists $a > 0$ for which $I(s, a) + b(a)^2 s < I(s, 0) + b(0)^2 s$, or

$$I(s, a) < \frac{a(2t - a)}{4s}.$$

For $a > 0$ sufficiently small we have $a(2t - a)/(4s) > -Ca$, where we can take $C = |t|/s$ if $t < 0$, $C = 1$ if $t = 0$, and $C = 0$ if $t > 0$. On the other hand, from (3.40), we have

$$I(s, a) \leq A_3 s a^{2/3} + A_2 a^{5/3} \leq A_3 s a^{2/3}.$$

Choosing $a > 0$ so small that $|A_3| s a^{2/3} > Ca$, we obtain the desired result. □

Lemma 4.4. *Suppose $s > 0$ and $t \in \mathbb{R}$, and define $b(a) = (a - t)/(2s)$ for $a \geq 0$. If $\{(h_n, g_n)\}$ is a minimizing sequence for $W(s,t)$, then there exist a positive number a and a subsequence $\{(h_{n_k}, g_{n_k})\}$ such that $\{(e^{ib(a)x} h_{n_k}, g_{n_k})\}$ is a minimizing sequence for $I(s,a)$, and*

$$W(s,t) = I(s,a) + b(a)^2 s. \tag{4.6}$$

Proof. For each $n \in \mathbb{N}$, define $a_n \geq 0$ by

$$a_n = \int_{-\infty}^{\infty} g_n^2 \; dx = G(h_n, g_n) + 2 \;\mathrm{Im} \int_{-\infty}^{\infty} h_n (\overline{h_n})_x \; dx.$$

Then a_n remains bounded by Lemma 4.1, so by passing to a subsequence we may assume that a_n converges to a limit $a \geq 0$. Let $b = b(a)$, and define $f_n(x) = e^{ibx} h_n(x)$. Then

$$\begin{aligned} \lim_{n\to\infty} E(f_n, g_n) &= \lim_{n\to\infty} \left(E(h_n, g_n) + b^2 H(h_n) - b \left(a_n - G(h_n, g_n) \right) \right) \\ &= W(s,t) + b^2 s - b(a - t) = W(s,t) - b^2 s \leq I(s,a), \end{aligned} \tag{4.7}$$

where we have used Lemma 4.2 and Lemma 4.3.

Next we claim that

$$\lim_{n\to\infty} E(f_n, g_n) \geq I(s,a). \tag{4.8}$$

In case $a > 0$, we prove (4.8) by defining $\beta_n = \sqrt{s}/\|f_n\|$ and $\theta_n = \sqrt{a}/\|g_n\|$, so that $\beta_n \to 1$ and $\theta_n \to 1$ as $n \to \infty$, and observing that $\lim_{n\to\infty} E(f_n, g_n) = \lim_{n\to\infty} E(\beta_n f_n, \theta_n g_n)$, while $E(\beta_n f_n, \theta_n g_n) \geq I(s,a)$ for all n. In case $a = 0$, we have $\|g_n\| \to 0$, and since $\|g_n\|_1$ and $\|f_n\|_1$ remain bounded by Lemma 4.1, it follows as in the proofs of Lemmas 3.10 and 3.25 that $\int_{-\infty}^{\infty} g_n^3 \; dx \to 0$ and $\int_{-\infty}^{\infty} g_n |f_n|^2 \; dx \to 0$. Therefore $\lim_{n\to\infty} E(f_n, g_n) \geq 0 = I(s,0)$, as desired.

It now follows from (4.7) and (4.8) that (4.6) holds, and that $E(f_n, g_n) \to I(s,a)$, which shows that $\{(f_n, g_n)\}$ is a minimizing sequence for $I(s,a)$. Finally, (4.6) and (4.3) imply that $a > 0$. □

Theorem 4.5. *Suppose $q \in (q_1, q_2)$, and let $s > 0$ and $t \in \mathbb{R}$ be given. Then every minimizing sequence $\{(h_n, g_n)\}$ for $W(s,t)$ is relatively compact in Y up to translations; i.e., there is a subsequence $\{(h_{n_k}, g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that*

$$(h_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k))$$

converges strongly in Y to some (h, g), which is a minimizer for $W(s,t)$.

Proof. By Lemma 4.4, given a minimizing sequence $\{(h_n, g_n)\}$ for $W(s,t)$, we may assume on passing to a subsequence that $\{e^{ibx} h_n(x), g_n(x)\}$ is a minimizing sequence for $I(s,a)$, where $a > 0$, $b = b(a)$, and (4.6) holds. Then Theorem 3.26 allows us to conclude, again after passing to a subsequence, that there exist numbers y_n such that

$$(e^{ib(x+y_n)} h_n(x + y_n), g_n(x + y_n))$$

converges in Y to some (f,g) which minimizes $I(s,a)$. By passing to a subsequence yet again, we may assume that $e^{iby_n} \to e^{i\theta}$ for some $\theta \in [0,2\pi)$. We then have that $(h_n(\cdot+y_n), g_n(\cdot+y_n)) \to (h,g)$ in Y, where $h(x) = e^{-i(bx+\theta)}f(x)$. Now Lemma 4.2 gives

$$\begin{aligned} I(s,a) = E(f,g) &= E(h,g) + b^2 H(h) - 2b \,\mathrm{Im} \int_{-\infty}^{\infty} h\overline{h}_x \, dx \\ &= E(h,g) + b^2 s + b(G(h,g) - \|g\|^2) \\ &= E(h,g) - b^2 s. \end{aligned} \tag{4.9}$$

From (4.6) and (4.9) we get $E(h,g) = W(s,t)$, so (h,g) is a minimizer for $W(s,t)$. □

As a consequence of Theorem 4.5, we can now assert the existence of a two-parameter family of ground-state solutions of (1.2), when $q \in (q_1, q_2)$. For $s > 0$ and $t \in \mathbb{R}$, define

$$F_{s,t} = \{(h,g) \in Y : E(h,g) = W(s,t),\ H(h) = s, \text{ and } G(h,g) = t\}.$$

From Theorem 4.5 we see, in particular, that $F_{s,t}$ is non-empty. In the next section we will see that $F_{s,t}$ is also stable.

Remark 4.6. It is natural to ask whether Theorem 3.27, which is valid for all $q > 0$, can be used to establish a result on ground-state solutions similar to Theorem 4.5. In fact, although Lemma 4.4 is valid for all $q > 0$, it turns out that one can not obtain a compactness result for minimizing sequences of $W(s,t)$ from Theorem 3.27 without a finer knowledge of the function $I(s,a)$. We do not pursue this topic here, and limit ourselves to stating an extra assumption which would lead to such a result. Suppose it could be shown that (4.6) uniquely defines a as a function of s and t. Then the above arguments allow us to deduce the following from Theorem 3.27: if (s_0,t_0) is such that, for some $\beta, \gamma > 0$,

$$I(s_0, a(s_0,t_0)) = \inf\{I(s,a) : \ s \geq 0,\ a \geq 0, \text{ and } s + \gamma a = \beta\},$$

then every minimizing sequence for $W(s_0,t_0)$ is relatively compact in Y up to translations. Moreover the set of minimizers for $W(s_0,t_0)$ is stable, in the sense of Theorem 5.4 below.

5. Ground-state solutions

We begin this section with a couple of results showing that the qualitative description of bound states in Theorem 2.1 can be improved when the solutions in question are ground states.

Theorem 5.1. *Suppose $s,t > 0$. If $(f,g) \in G_{s,t}$ then there exist $\sigma > 0$ and $c > 0$ such that* (2.2) *holds. Moreover, $g(x) > 0$ for all $x \in \mathbb{R}$, and there exist $\theta \in \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ such that $f(x) = \varphi(x)e^{i\theta}$ and $\varphi(x) > 0$ for all $x \in \mathbb{R}$.*

Proof. If $(f,g) \in G_{s,t}$, then by the Lagrange multiplier principle (cf. Theorem 7.7.2 of [27]), (f,g) is a solution of the Euler-Lagrange equation

$$\delta E(f,g) = \lambda\, \delta H(f,g) + \mu\, \delta H_1(f,g), \tag{5.1}$$

where H and H_1 are defined as operators on Y by $H(f,g) = \|f\|^2$ and $H_1(f,g) = \|g\|^2$, δ denotes the Fréchet derivative, and $\lambda, \mu \in \mathbb{R}$ are the Lagrange multipliers. Computing the Fréchet derivatives involved, we see that (5.1) becomes

$$\begin{cases} -f'' - gf = \lambda f \\ -2g'' - 3qg^2 - |f|^2 = 2\mu g, \end{cases} \tag{5.2}$$

which is (2.2) with $\sigma = -\lambda$ and $c = -2\mu$.

We claim that $\lambda < 0$ and $\mu < 0$. To see this, multiply the first equation in (5.2) by $\overline{f}$ and integrate over $\mathbb{R}$ to obtain that

$$\lambda s = K(f,g), \tag{5.3}$$

and multiply the second equation in (5.2) by g and integrate over $\mathbb{R}$ to obtain that

$$\mu t = \int_{-\infty}^{\infty} \left((g')^2 - \frac{1}{2} g|f|^2 - \frac{3}{2} qg^3 \right) \, dx \le \frac{1}{2} K(f,g) + \frac{3}{2} J(g). \tag{5.4}$$

Now from $I(s,t) = E(f,g)$ it follows that $K(f,g) = sM(g)$, and from the proof of parts (ii) and (iv) of Lemma 3.19, we see that $M(g) < 0$ and $J(g) < 0$. Therefore (5.3) and (5.4) imply that $\lambda < 0$ and $\mu < 0$.

We have now proved that (f,g) satisfies (2.2) with $\sigma > 0$ and $c > 0$. The remaining assertions of the theorem then follow from Theorem 2.1, except for the positivity of φ. To prove this, let $w = |\varphi|$, and observe that since $K(\varphi, g) = K(w,g) = sM(g)$ by Lemma 3.1, then (φ, g) and (w,g) are both in $G_{s,t}$. Hence, as shown above, we have

$$\begin{aligned} -\varphi'' - g\varphi &= \lambda \varphi, \\ -w'' - gw &= \lambda w \end{aligned} \tag{5.5}$$

where $\lambda = M(g)$. Multiplying the first equation in (5.5) by w and the second by φ and adding, we see that the Wronskian $W = \varphi w' - \varphi' w$ is constant. But since $W \to 0$ as $x \to \infty$ by Theorem 2.1, we must have $W(x) = 0$ for all $x \in \mathbb{R}$. Hence φ and w are linearly dependent, so φ must be of one sign on $\mathbb{R}$, and by changing the value of θ if necessary, we may assume that $\varphi(x) \ge 0$ on $\mathbb{R}$. Finally, since $\sigma = -\lambda > 0$, (5.5) implies that $K_\sigma * (g\varphi) = \varphi$, where K_σ is as defined in the proof of Theorem 2.1. It follows that $\varphi > 0$ on $\mathbb{R}$. □

Corollary 5.2. *Suppose $s > 0$ and $t \in \mathbb{R}$. If $(h,g) \in F_{s,t}$, then there exist $c > 0$ and $\omega > c^2/4$ such that (2.1) holds. Moreover, $g(x) > 0$ for all $x \in \mathbb{R}$, and there exist $\theta, b \in \mathbb{R}$ and $\varphi : \mathbb{R} \to \mathbb{R}$ such that $h(x) = e^{i\theta} e^{-ibx} \varphi(x)$ and $\varphi(x) > 0$ for all $x \in \mathbb{R}$.*

Proof. If $(h,g) \in F_{s,t}$, then as in the proof of Theorem 5.1, we have the Lagrange multiplier equation

$$\delta E(h,g) = \lambda \, \delta H(h,g) + \mu \, \delta G(h,g). \tag{5.6}$$

Computation of the Fréchet derivatives shows that (5.6) is equivalent to (2.1), with $\omega = -\lambda$ and $c = -2\mu$.

On the other hand, the sequence $\{(h_n, g_n)\}$ defined by $(h_n, g_n) = (h, g)$ for all $n \in \mathbb{N}$ is a minimizing sequence for $W(s, t)$, so from Lemma 4.4 it follows that $(e^{ibx}h(x), g(x)) \in G_{s,a}$, where $a > 0$ and $b \in \mathbb{R}$. Letting $f(x) = e^{ibx}h(x)$, we then have from Theorem 5.1 that (f, g) satisfies (2.2) for some $\sigma > 0$ and some $c > 0$. Substituting $f(x) = e^{ibx}h(x)$ into (2.2) and comparing with (2.1), we see that $b = -c/2$ and $\omega = \sigma + b^2 = \sigma + c^2/4$. Therefore $\omega > c^2/4$. The remaining assertions of the corollary follow immediately from Theorem 5.1. □

Next we show that the set $F_{s,t}$ is stable with regard to the flow generated by system (1.2). Concerning the well-posedness of (1.2), a variety of results have appeared, showing that (1.2) can be posed, at least locally in time, in Sobolev spaces of low order [7, 34]. For our purposes, the following result, due to Guo and Miao [21], is most convenient because it is set in the energy space Y.

Theorem 5.3. *Assume $q \neq 0$ in (1.2). Suppose $(\varphi, \psi) \in Y$. Then for every $T > 0$, (1.2) has a unique solution $(u, v) \in \mathcal{C}([0, T], Y)$ satisfying*

$$(u(x, 0), v(x, 0)) = (\varphi(x), \psi(x)).$$

The map $(\varphi, \psi) \mapsto (u, v)$ is a locally Lipschitz map from Y to $\mathcal{C}([0, T], Y)$. Moreover, $E(u(\cdot, t), v(\cdot, t))$, $G(u(\cdot, t), v(\cdot, t))$, and $H(u(\cdot, t))$ are independent of $t \in [0, T]$.

In particular, we note that the regularity result in Theorem 5.3 is enough to allow one to prove the invariance of the functionals E, G, and H along the solutions being considered. This may be done in the usual way, by first establishing the invariance of the functionals for smooth solutions, and then using the continuity of solutions with respect to their initial data to extend the result to solutions in $\mathcal{C}([0, T], Y)$. We omit the details of this argument.

Theorem 5.4. *Suppose $s > 0$ and $t \in \mathbb{R}$. For every $\epsilon > 0$ there exists $\delta > 0$ with the following property. Suppose $(\varphi, \psi) \in Y$ and*

$$\inf_{(h,g) \in F_{s,t}} \|(\varphi, \psi) - (h, g)\|_Y < \delta,$$

and let $(u(x, t), v(x, t))$ be the unique solution of (1.2) with

$$(u(x, 0), v(x, 0)) = (\varphi(x), \psi(x)),$$

guaranteed by Theorem 5.3 to exist in $\mathcal{C}([0, T], Y)$ for every $T > 0$. Then

$$\inf_{(h,g) \in F_{s,t}} \|(u(\cdot, t), v(\cdot, t)) - (h, g)\|_Y < \epsilon$$

for all $t \geq 0$.

Proof. Suppose that $F_{s,t}$ is not stable. Then there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$, we can find $(\varphi_n, \psi_n) \in Y$, and $t_n \geq 0$ such that

$$\inf_{(h,g) \in F_{s,t}} \|(\varphi_n, \psi_n) - (h, g)\|_Y < \frac{1}{n} \tag{5.7}$$

and

$$\inf_{(h,g)\in F_{s,t}} \|(u_n(\cdot,t_n),v_n(\cdot,t_n))-(h,g)\|_Y \geq \epsilon, \tag{5.8}$$

where $(u_n(x,t),v_n(x,t))$ solves (1.2) with initial data

$$(u_n(x,0),v_n(x,0)) = (\varphi_n(x),\psi_n(x)).$$

For brevity let us denote $u_n(\cdot,t_n)$ by Φ_n and $v_n(\cdot,t_n)$ by Ψ_n.

From (5.7) it follows that $\lim_{n\to\infty} E(\varphi_n,\psi_n) = W(s,t)$, $\lim_{n\to\infty} H(\varphi_n) = s$, and $\lim_{n\to\infty} G(\varphi_n,\psi_n) = t$. By Theorem 5.3, this implies that $\lim_{n\to\infty} E(\Phi_n,\Psi_n) = W(s,t)$, $\lim_{n\to\infty} H(\Phi_n) = s$, and $\lim_{n\to\infty} G(\Phi_n,\Psi_n) = t$. Therefore $\{(\Phi_n,\Psi_n)\}$ is a minimizing sequence for $W(s,t)$.

Now, by Theorem 4.5, there exists a subsequence $\{(\Phi_{n_k},\Psi_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that $(\Phi_{n_k}(\cdot+y_k),\Psi_{n_k}(\cdot+y_k))$ converges strongly in Y to some $(h_0,g_0)\in F_{s,t}$. In particular, there exists k large enough that

$$\|(\Phi_{n_k}(\cdot+y_k),\Psi_{n_k}(\cdot+y_k))-(h_0,g_0)\|_Y < \epsilon.$$

But this implies

$$\|(\Phi_{n_k},\Psi_{n_k})-(h_0(\cdot-y_k),g_0(\cdot-y_k))\|_Y < \epsilon,$$

and the invariance under translations of the functionals E, H, and G implies that $(h_0(\cdot-y_k),g_0(\cdot-y_k))$ is also in $F_{s,t}$. Therefore

$$\inf_{(h,g)\in F_{s,t}} \|(\Phi_{n_k},\Psi_{n_k})-(h,g)\|_Y < \epsilon,$$

contradicting (5.8). □

We conclude with a result on the ground-state solutions of (1.4). By definition, $(u(x,t),v(x,t))$ is a bound-state solution of (1.4) if u and v are of the form given by (1.7). Equivalently, h and g in (1.7) must satisfy the equations

$$\begin{cases} h''-\omega h - ich' = -hg \\ cg = |h|^2, \end{cases} \tag{5.9}$$

which are the Euler-Lagrange equations for the variational problem

$$W_1(s,t) = \inf\{K(h,g) : (h,g)\in X,\ H(h)=s, \text{ and } G(h,g)=t\}. \tag{5.10}$$

If we put $h(x)=e^{icx/2}f(x)$ in (5.9), we obtain the system

$$\begin{cases} f''-\sigma f = -fg \\ cg = |f|^2, \end{cases} \tag{5.11}$$

where $\sigma = \omega - \frac{c^2}{4}$. From Lemma 2.2, we see that the only solutions of (2.2) are given by $f(x) = e^{i\theta_0}f_1(x+x_0)$, $g(x) = g_1(x+x_0)$, where $\theta_0, x_0 \in \mathbb{R}$, and f_1, g_1 are as given in (2.12) with $\lambda = \sigma > 0$ and $\mu = c > 0$. Therefore these are all the bound-state solutions of (1.4).

In [24], Laurençot stated the following well-posedness result for (1.4).

Theorem 5.5. *For every $T > 0$ and every $(u_0, v_0) \in H^2_{\mathbb{C}} \times H^1$, there is a unique solution $(u(x,t), v(x,t))$ to (1.4) in $\mathcal{C}([0,T], H^2_{\mathbb{C}} \times H^1)$ such that $(u(x,0), v(x,0)) = (u_0, v_0)$. Moreover, the map from (u_0, v_0) to (u, v) is a continuous map from $H^2_{\mathbb{C}} \times H^1$ to $\mathcal{C}([0,T], H^2_{\mathbb{C}} \times H^1)$, and we have*

$$K(u(\cdot,t), v(\cdot,t)) = K(u_0, v_0)$$

for all $t \in [0,T]$.

Remark 5.6. Besides the preceding result, there have appeared several other well-posedness results for (1.4) in Sobolev spaces of low order [6, 8, 35, 36]. However, these results do not guarantee invariance of the energy functional K, which we need below. To our knowledge it remains an open question whether (1.4) is well-posed in the energy space X.

In the same paper, Laurençot established a stability result for bound- state solutions of (1.4). Here we recover Laurençot's stability result (see Theorem 5.7(iii)), and we also obtain the additional fact that the bound-state solutions of (1.4) are in fact ground states. That is, any critical point for the variational problem (5.10) is actually a global minimizer, or in other words, an element of the set

$$F^1_{s,t} = \{(h,g) \in X : K(h,g) = W_1(s,t),\ H(h) = s, \text{ and } G(h,g) = t\}.$$

for some $s > 0$ and $t \in \mathbb{R}$.

Theorem 5.7. *Suppose $s > 0$ and $t \in \mathbb{R}$. Then*

(i) every minimizing sequence $\{(h_n, g_n)\}$ for $W_1(s,t)$ is relatively compact in X up to translations; i.e., there is a subsequence $\{(h_{n_k}, g_{n_k})\}$ and a sequence of real numbers $\{y_k\}$ such that

$$(h_{n_k}(\cdot + y_k), g_{n_k}(\cdot + y_k))$$

converges strongly in X to some (h,g), which is a minimizer for $W_1(s,t)$.

(ii) in particular, $F^1_{s,t}$ is non-empty, and consists of all pairs (f,g) with $f(x) = e^{i\theta_0} f_1(x + x_0)$ and $g(x) = f_1(x + x_0)$, where $\theta_0, x_0 \in \mathbb{R}$, and f_1, g_1 are as given in (2.12) with $\lambda = (3t/16)^{2/3}$ and $\mu = s(12t)^{-1/3}$.

(iii) $F^1_{s,t}$ is stable, in the sense that for every $\epsilon > 0$ there exists $\delta > 0$ with the following property. Suppose $(\varphi, \psi) \in H^2_{\mathbb{C}} \times H^1$ and

$$\inf_{(h,g) \in F^1_{s,t}} \|(\varphi, \psi) - (h,g)\|_X < \delta,$$

and let $(u(x,t), v(x,t))$ be the unique solution of (1.4) with

$$(u(x,0), v(x,0)) = (\varphi(x), \psi(x)),$$

guaranteed by Theorem 5.5 to exist in $\mathcal{C}([0,T], H^2_{\mathbb{C}} \times H^1)$ for every $T > 0$. Then

$$\inf_{(h,g) \in F^1_{s,t}} \|(u(\cdot,t), v(\cdot,t)) - (h,g)\|_X < \epsilon$$

for all $t \geq 0$.

Proof. To prove (i), we need make only minor modifications to the proof of Theorem 4.5. In fact, the statements and proofs of Lemmas 4.1, 4.2, 4.3, and 4.4 continue to be valid if we replace throughout E by K, W by W_1, and I by I_1, except that we can use (3.10) instead of (3.40) at the end of Lemma 4.4. The statement and proof of Theorem 4.5 also remain valid once the same modifications are made, except that we use Theorem 3.12 instead of Theorem 3.26.

Since every ground state in $F^1_{s,t}$ is also a bound state, statement (ii) follows from (i) and the remarks concerning bound states which were made after (5.11).

Finally, the proof of (iii) is identical to that of Theorem 5.4, once the obvious modifications are made. □

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