STABILITY AND SYMMETRY OF SOLITARY-WAVE SOLUTIONS TO SYSTEMS MODELING INTERACTIONS OF LONG WAVES

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Abstract. We consider systems of equations which arise in modelling strong interactions of weakly nonlinear long waves in dispersive media. For a certain class of such systems, we prove the existence and stability of localized solutions representing coupled solitary waves travelling at a common speed. Our results apply in particular to the systems derived by Gear and Grimshaw and by Liu, Kubota, and Ko as models for interacting gravity waves in a density-stratified fluid. For the latter system, we also prove that any coupled solitary-wave solution must have components which are all symmetric about a common vertical axis.

1. Introduction.

Model equations for long, weakly nonlinear waves in fluids are typically derived by expanding the full equations of motion to first order in a small parameter ϵ determining the size of the wave amplitude and inverse wavelength. (The use of one small parameter to describe these two small quantities implicitly assumes a certain balance between them.) The solutions of the model equations describe the slow evolution, due to weak dispersive and nonlinear effects, of a wave which in the linear, non-dispersive limit corresponds to a mode of a linear eigenvalue problem.

The well-known Korteweg-de Vries equation, for example, was derived in this way by Benney [7] as a model for internal waves in a vertically stratified fluid. To zeroth order in ϵ, the full equations of motion are separable in the horizontal and vertical space coordinates, giving rise to a Sturm-Liouville problem in the vertical coordinate and the linear wave equation $u_{tt} - c_j^2 u_{xx} = 0$ in the horizontal coordinate, where the wavespeed c_j corresponds to the jth eigenvalue of the Sturm-Liouville problem. Each individual eigensolution of the Sturm-Liouville problem thus gives rise to a wave with fixed vertical structure and horizontal speed $\pm c_j$. The Korteweg-de Vries equation describes the effects of weak nonlinearity and weak dispersion on such a wave in the case when the horizontal motion is unidirectional.

In this paper, we consider systems of equations which have been derived as models for the interaction of two (or more) long waves, each of which corresponds to a different underlying mode or vertical structure. Such systems generally take the form

$$h_t + D^{-1}(\nabla N(h) - Lh)_x = 0, \tag{1.1}$$

where h is an $\mathbb{R}^n$-valued function of x and t, D is a positive $n \times n$ diagonal matrix, ∇N is the gradient of a homogeneous function $N : \mathbb{R}^n \to \mathbb{R}$, and L is a Fourier multiplier operator which acts self-adjointly on the Sobolev space in which (1.1) is posed. The interesting case is when the wavespeeds corresponding to two different modes have nearly the same value, so that the modes interact on a time scale long enough for nonlinearity and dispersion to have a significant effect.

In particular we are interested in solitary waves, or localized travelling-wave solutions of (1.1) of permanent form. More precisely, by a solitary wave we mean a function $g(x) = (g_1(x), \dots, g_n(x))$ such that $g_1, \dots, g_n$ are in $L^2(\mathbb{R})$ and $h(x,t) = g(x - ct)$ is a solution of (1.1), for some real number c. In the scalar case $n = 1$ (which includes the Korteweg-de Vries equation as a specific example), it is well known that such waves often play an important or even dominant role in the evolution of general solutions of nonlinear dispersive wave equations (cf. [9]). This is due in large part to the remarkable stability properties of solitary waves, which enable them to retain their identity even under large perturbations. Theoretical explanations of the stability of solitary-wave solutions of (1.1) have undergone active development in the past three decades, but has so far been restricted to the scalar case (for a brief overview and some references, see [2]). It is our intention here to extend some of this work to the case $n > 1$.

The approach we take to stability theory here is the same that has underlain all proofs of stability of solitary waves (dating back to one given by Boussinesq himself in 1872 [12]). First, we observe that equation (1.1) can be put in Hamiltonian form, and hence has the Hamiltonian functional E itself as a conserved functional. Another conserved functional Q is defined by $Q(h) = \int_{-\infty}^{\infty} \frac{1}{2}\langle h, Dh\rangle \; dx$. It turns out (see Section 2 below) that g is a solitary-wave solution of (1.1) if and only if g is a critical point for the constrained variational problem of minimizing E over a level set of Q. Moreover, a standard argument shows that if g is actually a local minimizer for E under this constraint, then G, the intersection of the level sets of E and Q containing g, is a stable set of solitary waves. This means that for every $\epsilon > 0$, there exists $\delta > 0$ such that if h is within δ of G (in an appropriate norm) at time $t = 0$, then h remains within ϵ of G for all times $t \geq 0$.

In Theorem 2.1 below, we give sufficient conditions for the existence of stable sets of solitary-wave solutions of (1.1). The conditions include one which is phrased in terms of the above-mentioned variational problem, but as pointed out in Theorem 2.2, in some important situations all the conditions can be reduced to simple properties of the function N and the symbol of the operator L. The proof of Theorem 2.1, which is given in Section 3, proceeds by using P. Lions' method of concentration compactness to show the existence of a non-empty set of global minimizers of E on each level set of Q. The use of concentration compactness to prove existence and stability of solitary waves goes back to a paper of Cazenave and Lions on the nonlinear Schrödinger equation [13], and has since been developed by a number of authors (see, e.g., [3,6,14,18,31]. Our point of departure is the method of [2], which was easily adapted to handle the systems considered here.

In Section 4, we apply Theorems 2.1 and 2.2 to prove the existence of stable sets of solitary-wave solutions to systems modelling the strong interaction of long internal waves in stratified fluids. In the first of these systems, derived by Gear and Grimshaw in [20], the components $h_1(x,t)$ and $h_2(x,t)$ of $h(x,t)$ represent the slow horizontal variations, due to weak nonlinearity and dispersion, of two long waves which in the linear, non-dispersive limit correspond to two different vertical modes. In the other system, derived by Liu, Kubota, and Ko in [27], h_1 and h_2 represent small, long-wavelength disturbances at two pycnoclines separated by a region of constant density. (The question of how exactly the situations governed by the two systems relate to each other physically is an interesting one, to which the present authors do not yet know the answer. In particular, there is no way to obtain one equation as a scaling limit of the other.)

For reasons mentioned earlier, the derivations of both systems assume that the waves represented by h_1 and h_2 travel at nearly the same speed. It is also possible to derive systems with $n \geq 3$, describing the strong simultaneous interactions of three or more underlying modes, and Theorem 2.1 applies to such systems as well. However, these systems are of limited physical interest, since in a given fluid it is relatively unlikely that one can find three linear modes whose corresponding wavespeeds are close enough for such interactions to occur.

We note that an existence result for Liu-Kubota-Ko solitary waves appears in [3], and an existence result for Gear-Grimshaw solitary waves appears in [10]. Both these papers use the concentration compactness technique to obtain solitary waves as global minimizers to constrained variational problems. However, since the minimized functional and the constraint functional are not constants of the motion, these results do not yield the stability of the solitary waves which are found to exist.

One motivation for the present study was provided by the numerical experiments conducted in [20] and [27]. Interestingly, Liu, Kubota, and Ko did not observe anything close to a steady travelling-wave solution of their system: instead, they found "leap-frog" solutions in which localized disturbances in h_1 and h_2 took turns overtaking and falling behind each other. Gear and Grimshaw, on the other hand, found that for typical values of the parameters in their equation, general initial data would quickly give rise to steady travelling-wave solutions which maintained their identity even after colliding with each other. They also were able to duplicate the leap-frogging behavior observed in [27] by choosing their parameters so as to decouple the nonlinear terms in their system. The stability results in the present paper validate the numerical observations of stable solitary waves made by Gear and Grimshaw, and also show that the observed leap-frog solutions do not arise due to lack of stability of solitary waves.

One issue which our stability result does not resolve, however, is that of the structure of the stable sets of solitary waves. Indeed, this is a general drawback of the concentration-compactness approach to stability as compared with other approaches involving finer analysis (cf. the discussion in [2]). This issue has bearing on the leap-frog solutions mentioned in the preceding paragraph: if, for example, it were the case that the stable set of solitary waves included functions $g = (g_1, g_2)$ such that the maxima of g_1 and g_2 are located at different points on the x-axis, then a leap-frog solution might actually represent a solution which stays at all times very close to the stable set.

To shed light on this latter question, we investigate the symmetry properties of solitary-wave solutions to the Liu-Kubota-Ko system in Section 5. In Theorem 5.4 we show that, in case the coefficients of the nonlinear terms in the system are positive, then the solitary waves in the stable sets found in Section 4 must have components which are both symmetric about the same value of x and which decay monotonically to zero away from their common axis of symmetry. Hence, if a leap-frog solution exists in this case, it cannot be said to closely resemble a solitary wave at any given time. This would suggest that while solitary waves are stable in the sense of Theorem 2.1, they may not be *asymptotically* stable in the sense of Lyapunov. (This would contrast with the strong asymptotic stability properties of KdV solitary waves [30].) Unfortunately, there remains a gap in the evidence: since the leap-frog solutions observed in [27] were for a system in which the coefficients of the nonlinear term were of mixed sign, it is not clear yet whether such solutions exist in the case of positive coefficients. On the other hand, we note that the leap-frog solutions observed in [20] were obtained in the case in which the coefficients of the nonlinear terms were both positive.

Theorem 5.4 is actually closely related to a result of Maia [29] for the full equations of motion of an incompressible, inviscid stratified fluid. Maia's work in turn represents a development of the symmetry theory for solitary waves initiated by Craig and Sternberg [16,17], in particular incorporating into the arguments of [17] some simplifications suggested by the work of Congming Li [25]. Our proof essentially follows the lines of Maia's, with some modifications and further simplifications appropriate to the present situation.

Finally, we also obtain, in Theorems 5.5 and 5.6, a monotonicity result for bore-like solutions to the problem modeled by the Liu-Kubota-Ko system, and a symmetry result for solitary-wave solutions of a scalar equation derived by Kubota, Ko, and Dobbs [24] as a model for long internal waves in a stratified fluid.

A preliminary version of Theorem 2.1 was announced in [4].

Notation. We use $\langle \cdot, \cdot \rangle$ to denote the usual inner product in $\mathbb{C}^n$; i.e., for $v = (v_1, \dots, v_n)$ and $w = (w_1, \dots w_n)$ in $\mathbb{C}^n$ we set $\langle v, w \rangle = v_1 \overline{w_1} + \cdots + v_n \overline{w_n}$, where bars denote complex conjugation. For v in $\mathbb{C}^n$ (or in $\mathbb{R}^n$) we define $|v| = \langle v, v \rangle^{1/2}$.

Let I be an interval in $\mathbb{R}$. As usual, for $1 \le p < \infty$, $L^p(I)$ denotes the set of all measurable functions $f : \mathbb{R} \to \mathbb{R}$ such that $(\int_{-\infty}^{\infty} |f(x)|^p \, dx)^{1/p} < \infty$. We define $X_p(I)$ to be the Banach space of all measurable functions $f : I \to \mathbb{R}^n$ such that $|f|_{X_p(I)} < \infty$, where

$$|f|_{X_p(I)} = \left(\int_I |f(x)|^p \, dx \right)^{1/p}.$$

For $s \in \mathbb{R}$, let $H_s(I)$ denote the L^2-based Sobolev space of order s on I, and define $Y_s(I) = (H_s(I))^n = \{f = (f_1, \dots, f_n) : f_i \in H_s(I) \text{ for } i = 1, \dots, n\}$, with norm given by

$$\|f\|_{Y_s(I)} = \|f_1\|_{H_s(I)} + \cdots + \|f_n\|_{H_s(I)}.$$

In case $I = \mathbb{R}$, the spaces $X_p(I)$ and $Y_s(I)$ will be denoted by X_p and Y_s, and the corresponding norms will be denoted by $|f|_p$ and $\|f\|_s$. We define Y_∞ to be the intersection of all the spaces Y_s as s ranges over the set of all real numbers.

If H is any Hilbert space then $l_2(H)$ will denote the Hilbert space of all infinite sequences $x = (x_1, x_2, \dots)$, $x_i \in H$, such that

$$\|x\|_{l_2(H)} = \left(\sum_{j=1}^{\infty} \|x_j\|_H^2 \right)^{1/2} < \infty.$$

If Ω is any open subset of $\mathbb{R}^n$, $C^k(\Omega)$ denotes the set of all functions on Ω whose partial derivatives up to order k exist on Ω, and $C^k(\bar{\Omega})$ denotes the set of all functions whose partial derivatives up to order k exist on Ω and can be continuously extended to $\bar{\Omega}$. We also define $C^\infty(\Omega) = \cap_{k=0}^{\infty} C^k(\Omega)$ and $C^\infty(\bar{\Omega}) = \cap_{k=0}^{\infty} C^k(\bar{\Omega})$.

If X is a Banach space and G is a subset of X, we say that a sequence $\{x_n\}$ in X converges to G if

$$\lim_{n\to\infty} \inf_{g \in G} \|x_n - g\|_X = 0.$$

Also, for each $T > 0$, $C([0,T]; X)$ will denote the Banach space of all continuous maps h from $[0,T]$ to X, with norm defined by $\|h\|_{C([0,T];X)} = \sup_{t \in [0,T]} \|h(t)\|_X$.

Hats will always denote Fourier transforms with respect to x: $\widehat{\zeta}(k) = \int_{-\infty}^{\infty} e^{ikx} \zeta(x) \, dx$, where the integral is interpreted in the usual way for vector-valued functions ζ.

2. Sufficient conditions for stability of solitary waves.

Consider a vector-valued nonlinear dispersive wave equation of the form

$$h_t + D^{-1}(\nabla N(h) - Lh)_x = 0, \tag{2.1}$$

in which the unknown h is an $\mathbb{R}^n$-valued function of the variables x and t. The operators D, ∇N, and L in (2.1) are defined as follows:

- D is an $n \times n$ diagonal matrix with positive entries β_i along the diagonal.
- ∇N is the gradient of a function $N : \mathbb{R}^n \to \mathbb{R}$. We assume that N is homogeneous of degree $p+2$, where p is any positive number; or, in other words,

$$N(\theta v) = \theta^{p+2} N(v)$$

for every $v \in \mathbb{R}^n$ and every $\theta > 0$. Further, we require N to be twice continuously differentiable on the unit sphere Σ in $\mathbb{R}^n$ (and hence everywhere on $\mathbb{R}^n$). In particular, it follows from our assumptions on N that

$$\left| \int_{-\infty}^{\infty} N(f)\, dx \right| \leq C|f|_{p+2}^{p+2}$$

for all $f \in X_{p+2}$, where C is independent of f: to see this, notice that $|N(f)| = |f|^{p+2} N(f/|f|) \leq C|f|^{p+2}$, where C is the supremum of N on Σ.

- The dispersion operator L is a matrix Fourier multiplier operator defined by

$$\widehat{Lh}(k) = A(k)\widehat{h}(k)$$

for $k \in \mathbb{R}$, where $A(k)$, the symbol of L, is for each $k \in \mathbb{R}$ a symmetric $n \times n$ matrix with real entries, and $A(k)$ satisfies $A(-k) = A(k)$ for all $k \in \mathbb{R}$.

Further, we make the following assumptions on $A(k)$:

(A1) There exist positive constants C_1, C_2 and a number $s > p/4$ such that

$$C_1 |k|^{2s} |v|^2 \leq \langle A(k)v, v \rangle \leq C_2 |k|^{2s} |v|^2$$

for all vectors v in $\mathbb{R}^n$ and all sufficiently large values of $|k|$.

(A2) For each i and j between 1 and n, the matrix components $a_{ij}(k)$ are four times differentiable on $\{k \neq 0\}$. Moreover, there exist constants C and K such that for all $m \in \{1, 2, 3, 4\}$,

$$\left| \left(\frac{d}{dk} \right)^m \left(\frac{a_{ij}(k) - a_{ij}(0)}{k} \right) \right| \leq C|k|^{-m} \quad \text{for} \quad 0 < |k| \leq K,$$

and
$$\left|\left(\frac{d}{dk}\right)^m\left(\frac{\sqrt{|a_{ij}(k)|}}{k^s}\right)\right| \leq C|k|^{-m} \quad \text{for} \quad |k| \geq K.$$

We remark that the condition in assumption (A2) on the behavior of a_{ij} near the origin is satisfied whenever the derivatives up to order five of $a_{ij}(k)$ exist and are bounded on $(0, K]$.

A somewhat stronger condition on $A(k)$, which implies both (A1) and (A2) and has the advantage of being more convenient to verify, is the following:

(A3) The symmetric matrix $A(k)$ has n distinct eigenvalues $\lambda_1(k), \ldots, \lambda_n(k)$ which, together with their derivatives up to order five, are bounded on $0 < k < 1$ and continuous on $0 < k < \infty$. Furthermore, there exist positive constants C_1, C_2, and K such that for $1 \leq i \leq n$ and $0 \leq m \leq 4$ one has
$$C_1|k|^{2s-m} \leq \left(\frac{d}{dk}\right)^m \lambda_i(k) \leq C_2|k|^{2s-m} \quad \text{for} \quad |k| > K.$$

That (A3) implies (A1) and (A2) follows from the perturbation theory expounded in chapter II of [23] (see in particular Section II.5.3). Note also that if the $\lambda_i(k)$ are assumed to be analytic functions of k for $k > 0$, then the assumption that the eigenvalues are distinct may be dropped (cf. Theorem II.1.10 of [23]).

We will assume in what follows that equation (2.1) is globally well-posed in Y_r for some $r \geq s$. In other words, we assume that for every $h_0 \in Y_r$ and every $T > 0$, there exists a unique weak solution h of (2.1) in $C([0, T]; Y_r)$, and the correspondence $h_0 \mapsto h$ defines a continuous map from Y_r to $C([0, T]; Y_r)$. Here "weak solution" means any element h of $C([0, T]; Y_r)$ such that for all $t \geq 0$, h_t exists in Y_r (in the usual sense of the derivative of a Banach-space valued function), and is equal to $-D^{-1}(\nabla N(h) - Lh)_x$. Notice that our assumption on N guarantees that, for fixed t, $\nabla N(h(t))$ is in $L^{2/(p+1)}(\mathbb{R})$, and hence that $-D^{-1}(\nabla N(h) - Lh)_x$ exists as a tempered distribution on $\mathbb{R}$, so that the equality has sense.

In particular, we are concerned with *solitary-wave solutions* of (2.1), which by definition are solutions of the form $h(t) = \phi(\cdot - ct)$, where $\phi \in Y_r$ and c is a real number called the *wavespeed* of the solitary wave. We also refer to the profile ϕ itself as a solitary wave. Thus $\phi \in Y_r$ is a solitary wave if and only if it satisfies the equation

$$-cD\phi = L\phi - \nabla N(\phi). \tag{2.2}$$

We now define functionals Q and E on Y_s which are constants of the motion for (2.1) and which play a crucial role in the stability theory for solitary-wave solutions. Let

$$Q(f) = \int_{-\infty}^{\infty} \frac{1}{2}\langle f, Df\rangle \; dx$$

and

$$E(f) = \int_{-\infty}^{\infty} \frac{1}{2}\langle f, Lf\rangle - N(f) \; dx.$$

We claim that if h is a solution of (2.1) in $C([0,T];Y_r)$ then $Q(h(x,t))$ and $E(h(x,t))$ are independent of t. Indeed, taking the inner product of (2.1) with Dh and integrating over $\mathbb{R}$, one sees that $\frac{d}{dt}Q(h(x,t)) = 0$, at least if h is in $C([0,T];Y_{r'})$ for r' sufficiently large. Hence $Q(h(x,t)) = Q(h(x,0))$ for all t if h is a solution in $C([0,T];Y_{r'})$, and the result for solutions h in $C([0,T];Y_r)$ then follows from the assumed well-posedness properties of (2.1) and the fact that $Y_{r'}$ is dense in Y_r. Next, observe that (2.1) may be written in Hamiltonian form as

$$h_t = J\,\delta E(h),$$

where δE denotes the Fréchet derivative of E and

$$J = \partial_x D^{-1}$$

is antisymmetric with respect to the inner product in Y_r. It follows that E plays the role of a Hamiltonian functional for (2.1), and in particular is a constant of the motion.

The importance of the functionals E and Q for our purposes rests on the fact that (2.2) can be written in the form

$$\delta E(\phi) = -c\,\delta Q(\phi). \tag{2.3}$$

We will show that, under the assumptions stated below in Theorem 2.1, the problem of minimizing E subject to constant Q always has a non-empty solution set. But since each element of the solution set must satisfy the Euler-Lagrange equation (2.3), the solution set must consist of solitary waves.

Actually, in what follows it will be more convenient to work with a modified functional E_0 than with the functional E defined above. To define E_0, first consider the operator $\sigma D + L$, where σ ranges over the set of real numbers. From the perturbation theory of symmetric matrices (see Theorem II.6.8, p. 122 of [23]), it follows that there exist n functions $\lambda_1(k,\sigma), \dots, \lambda_n(k,\sigma)$, representing the (unordered, and possibly repeated) eigenvalues of $\sigma D + A(k)$, which depend smoothly on σ and, for a given σ, have the same differentiability and continuity properties with respect to k as do the functions $a_{ij}(k)$.

From the variational characterization of eigenvalues, we have that the least eigenvalue of $\sigma D + A(k)$ is the infimum of the set of values of $\langle(\sigma D + A(k))v, v\rangle$ as v ranges over the set of vectors in $\mathbb{R}^n$ such that $\|v\| = 1$. It follows easily that the function $b(\sigma)$ defined by

$$b(\sigma) = \inf \{\lambda_i(k,\sigma) : 0 \le k < \infty \quad \text{and} \quad 1 \le i \le n\}$$

is a strictly decreasing function of σ. Moreover, since

$$b(\sigma) \ge \sigma \left(\min_{1 \le i \le n} \beta_i \right) + b(0),$$

and $b(0) > -\infty$ as a consequence of (A1) and (A2), then $b(\sigma) > 0$ for σ sufficiently large. Also, since for any given k one has

$$b(\sigma) \le \sup_{\|v\|=1} \langle(\sigma D + A(k))v, v\rangle \le \sigma \left(\max_{1 \le i \le n} \beta_i \right) + \max_{1 \le i \le n} \lambda_i(k, 0),$$

it follows that $b(\sigma) < 0$ for σ sufficiently large and negative. We conclude that there exists a unique σ_0 such that $b(\sigma_0) = 0$.

The number σ_0 can be characterized as the smallest value of σ such that the matrix $\sigma D + A(k)$ is non-negative for all $k \in \mathbb{R}$. Alternatively, we can view σ_0 as the greatest possible eigenvalue of $-D^{-1}A(k)$, as k ranges over $\mathbb{R}$. Hence σ_0 is the largest possible wavespeed of infinitesimal sinusoidal waves, i.e., σ_0 is the largest value of σ such that the linearized equation

$$h_t - D^{-1}(Lh)_x = 0$$

has a solution of the form $h(x,t) = ve^{ik(x-\sigma t)}$ with nonzero $v \in \mathbb{R}^n$.

We now define $\Lambda = \sigma_0 D + L$, and define the functional E_0 by

$$E_0(f) = \int_{-\infty}^{\infty} \frac{1}{2}\langle f, \Lambda f\rangle - N(f)\ dx,$$

so that $E_0 = \sigma_0 Q + E$. Notice that replacing L by $\sigma_0 D + L$ in (2.1) amounts to nothing more than changing to new coordinates x' and t' given by $x' = x - \sigma_0 t$ and $t' = t$. Thus, up to a Galilean coordinate change, one can always assume that $\Lambda = L$ and $E_0 = E$.

Define the number I_q by

$$I_q = \inf\{E_0(f) : f \in Y_s \text{ and } \ Q(f) = q\}$$

The set of minimizers for I_q is

$$G_q = \{g \in Y_s : E_0(g) = I_q \text{ and } \ Q(g) = q\},$$

and the Euler-Lagrange equation for the constrained minimization problem solved by the functions in G_q is

$$\delta E_0(g) = \delta E(g) + \sigma_0\, \delta Q(g) = \lambda\, \delta Q(g),$$

where λ is the Lagrange multiplier. Comparing this equation with (2.3), we see that if $g \in G_q$, then g is a solitary-wave solution of (2.1) with wavespeed $c = \sigma_0 - \lambda$. (Notice that the multiplier λ could, in principle, vary from one element of G_q to the next).

We can now state the following result, giving a sufficient condition for the existence of a stable set of solitary-wave solutions of (2.1).

Theorem 2.1. *Suppose that s, p, and L are such that (A1) and (A2) hold. If the solution I_q of the variational problem defined above satisfies $I_q < 0$ for all $q > 0$, then for each $q > 0$ the set G_q of minimizers for the variational problem is non-empty, and each $g \in G_q$ is a solitary-wave solution of (2.1) with wavespeed $c > \sigma_0$. Moreover, G_q is a stable set of initial data for (2.1), in the following sense: for every $\epsilon > 0$ there exists δ such that if $h_0 \in Y_r$ and*

$$\inf_{g \in G_q} \|h_0 - g\|_s < \delta,$$

then the solution $h(x,t)$ of (2.1) with $h(x,0) = h_0$ satisfies

$$\inf_{g \in G_q} \|h(x,t) - g\|_s < \epsilon$$

for all $t \in \mathbb{R}$.

The proof of Theorem 2.1 is given in Section 3 below.

The next result, which is a corollary of Theorem 2.1, will apply to the model equations considered in Section 4.

Theorem 2.2. *Suppose that* s, p, *and* L *are such that (A1) and (A2) hold. Suppose also that there exists a vector* $v_0 \in \mathbb{R}^n$ *such that* $N(v_0) > 0$ *and*

$$|\langle v_0, (\sigma_0 D + A(k))v_0\rangle| \leq C|k|^{s_0} \quad \text{for all } |k| \leq 1, \tag{2.4}$$

where $C > 0$ *and* $s_0 > p/2$. *Then for each* $q > 0$, *the set* G_q *is non-empty and the elements* g *of* G_q *are solitary waves with wavespeeds* c *greater than* σ_0. *Moreover,* G_q *is stable in the sense of Theorem 2.1.*

Remarks.

(i) In particular, inequality (2.4) holds in the important special case when $\sigma_0 D + A(k)$ has the eigenvalue 0 at $k = 0$. This may be seen by taking v_0 to be an eigenvector for the eigenvalue 0 of $\sigma_0 D + A(0)$; then $\langle v_0, (\sigma_0 D + A(k))v_0\rangle$ defines a function of k which has the value 0 at $k = 0$ and has bounded derivative on $0 \leq k \leq 1$, and it follows that (2.4) holds for s_0 at least 1.

(ii) If $N(-v) = -N(v)$ for $v \in \mathbb{R}^n$, then we can drop the condition that $N(v_0) > 0$, since v_0 may be replaced by $-v_0$ if necessary.

Proof. We claim that the existence of a vector v_0 with the stated properties implies that $I_q < 0$ for each $q > 0$. To see this, let $w(x) = v_0\phi(x)$, where $\phi(x)$ is any non-negative smooth function with compact support, normalized so that $Q(w) = q$. For any $\theta > 0$ let $w_\theta(x) = \sqrt{\theta}\, w(\theta x)$. Then by assumption there exists a constant C such that for $|k| \leq 1$ and $\theta < 1/K$, where K is as in (A2),

$$\begin{aligned}
\left|\int_{-\infty}^{\infty} \langle w_\theta, \Lambda w_\theta\rangle\, dx\right| &= \frac{1}{\theta}\int_{-\infty}^{\infty} \langle v_0, (\sigma_0 D + A(k))v_0\rangle |\widehat{\phi}(k/\theta)|^2 dk \\
&= \int_{-\infty}^{\infty} \langle v_0, (\sigma_0 D + A(k\theta))v_0\rangle |\widehat{\phi}(k)|^2\, dk \\
&\leq C\theta^{s_0}\int_{|k|\leq 1/\theta} |k|^{s_0}|\widehat{\phi}(k)|^2\, dk + C\theta^{2s}\int_{|k|\geq 1/\theta} |k|^{2s}|\widehat{\phi}(k)|^2\, dk.
\end{aligned}$$

But because ϕ is smooth with compact support, $\widehat{\phi}(k)$ decays more rapidly than any power of k as $|k| \to \infty$, and it follows that the last integral in the preceding expression vanishes more rapidly than any power of θ as $\theta \to 0$. Therefore

$$\left| \int_{-\infty}^{\infty} \langle w_\theta, \Lambda w_\theta \rangle \; dx \right| \le C\theta^{s_0},$$

for all small values of θ, where C is independent of θ.

On the other hand,

$$\begin{aligned} \int_{-\infty}^{\infty} N(w_\theta) \; dx &= \theta^{p/2} \int_{-\infty}^{\infty} N(w) \; dx \\ &= \theta^{p/2} N\left(\frac{v_0}{|v_0|}\right) |v_0|^{p+2} \int_{-\infty}^{\infty} \phi(x)^{p+2} \; dx = C\theta^{p/2}, \end{aligned}$$

where $C > 0$ is independent of θ.

We conclude that in the expression

$$E_0(w_\theta) = \int_{-\infty}^{\infty} \frac{1}{2} \langle w_\theta, \Lambda w_\theta \rangle \; dx - \int_{-\infty}^{\infty} N(w_\theta) \; dx$$

the second integral on the right-hand side is positive and (since $s_0 > p/2$) goes to zero more slowly than the first term as $\theta \to 0$. It follows that $E_0(w_\theta) < 0$ for θ sufficiently near zero. On the other hand, one has $Q(w_\theta) = q$ for all θ. Therefore I_q must be less than zero, as claimed.

The conclusion of Theorem 2.2 now follows from Theorem 2.1. $\square$

3. Proof of Theorem 2.1.

The proof of Theorem 2.1 proceeds via P. Lions' method of concentration compactness [26], and follows the lines of the proof of stability of ground-state solutions of the nonlinear Schrödinger equation given by Cazenave and Lions in [13].

We begin with the following standard estimate.

Lemma 3.1. *Suppose I is an interval in $\mathbb{R}$, $p > 0$, and $s > p/4$. Then there exists $C > 0$ such that for all $f \in Y_s(I)$,*

$$|f|_{X_{p+2}(I)}^{p+2} \le C \|f\|_{Y_s(I)}^{p/2s} |f|_{X_2(I)}^{p+2-(p/2s)}.$$

Proof. Let $s' = \frac{p}{2(p+2)}$. From the Sobolev embedding theorem, it follows that there exists a constant C independent of f such that for all $f \in Y_{s'}(I)$,

$$|f|_{X_{p+2}(I)} \leq C\|f\|_{Y_{s'}(I)}.$$

The stated result then follows from the interpolation inequality

$$\|f\|_{Y_{s'}(I)} \leq C\|f\|_{Y_s(I)}^{s'/s}\|f\|_{Y_0(I)}^{1-(s'/s)},$$

since $Y_0(I) = X_2(I)$. $\square$

Lemma 3.2. *For all $q > 0$, we have $I_q > -\infty$.*

Proof. Let f be an arbitrary element of Y_s satisfying $Q(f) = q$; we wish to show that $E_0(f)$ is bounded below by a number which is independent of f.

From assumption (A1) and the definition of σ_0, it follows that there exist constants $C_3 > 0$ and $C_4 > 0$ such that

$$C_3(1+|k|)^{2s}|v|^2 \leq \langle v, [(\sigma_0+1)D + A(k)]\, v\rangle \leq C_4(1+|k|)^{2s}|v|^2$$

for all $k \in \mathbb{R}$ and $v \in \mathbb{C}^2$. Therefore the expression

$$\left(\int_{-\infty}^{\infty} \frac{1}{2}\langle f(x), \Lambda f(x)\rangle\, dx + Q(f)\right)^{1/2}$$

defines a norm on Y_s equivalent to $\|f\|_s$. In particular, it follows that we can write

$$\begin{aligned} E_0(f) &= E_0(f) + Q(f) - Q(f) \\ &= \int_{-\infty}^{\infty} \frac{1}{2}\langle f(x), \Lambda f(x)\rangle\, dx + Q(f) - \int_{-\infty}^{\infty} N(f)\, dx - Q(f) \\ &\geq C_3\|f\|_s^2 - C|f|_{p+2}^{p+2} - q, \end{aligned}$$

where C is a positive constant which is independent of f. But by Lemma 3.1 and Young's Inequality,

$$|f|_{p+2}^{p+2} \leq C\|f\|_s^{p/2s}|f|_2^{p+2-(p/2s)} \leq \epsilon\|f\|_s^2 + C|f|_2^{2+4sp/(4s-p)},$$

where $\epsilon > 0$ can be chosen arbitrarily small, and again C denotes various constants which may depend on ϵ but do not depend on f. Combining with the preceding estimate, and taking $\epsilon < C_3$, we now obtain

$$E_0(f) \geq -C|f|_2^{2+4sp/(4s-p)} - q.$$

The proof concludes with the observation that $|f|_2$ is dominated by a constant times $Q(f)$, and hence remains bounded due to the assumption that $Q(f) = q$. $\square$

We define a minimizing sequence for I_q to be any sequence $\{f_n\}$ of functions in Y_s satisfying

$$Q(f_n) = q \ \text{ for all } n$$

and

$$\lim_{n\to\infty} E_0(f_n) = I_q.$$

To each minimizing sequence $\{f_n\}$ is associated a sequence of nondecreasing functions $M_n : [0, \infty) \to [0, q]$ defined by

$$M_n(r) = \sup_{y\in\mathbb{R}} \int_{y-r}^{y+r} \frac{1}{2}\langle f_n, Df_n\rangle\, dx.$$

A standard argument shows that any uniformly bounded sequence of nondecreasing functions on $[0, \infty)$ must have a subsequence which converges pointwise to a nondecreasing limit function on $[0, \infty)$. Hence $\{M_n\}$ has such a subsequence, which we denote again by $\{M_n\}$. Let $M : [0, \infty) \to [0, q]$ be the nondecreasing function to which M_n converges, and define

$$\alpha = \lim_{r\to\infty} M(r),$$

so $0 \le \alpha \le q$.

Lemma 3.3. *If $\{f_n\}$ is a minimizing sequence for I_q, then there exist constants $B > 0$ and $\delta_2 > 0$ such that*

- *(i) $\|f_n\|_s \le B$ for all n and*
- *(ii) $\int_{-\infty}^{\infty} N(f_n)\, dx \ge \delta_2$ for all sufficiently large n.*

Proof. As was noted in the proof of Lemma 3.2, the quantity

$$\left(\int_{-\infty}^{\infty} \frac{1}{2}\langle f(x), \Lambda f(x)\rangle\, dx + Q(f)\right)^{1/2}$$

defines a norm on Y_s equivalent to $\|f\|_s$. Therefore

$$\begin{aligned}
\|f_n\|_s^2 &\le C\left(\int_{-\infty}^{\infty} \frac{1}{2}\langle f_n(x), \Lambda f_n(x)\rangle\, dx + Q(f_n)\right) \\
&\le C\left(\sup_n E_0(f_n) + |f_n|_{p+2}^{p+2} + q\right) \\
&\le C\big(1 + |f_n|_2^{p+2-(p/2s)} \|f_n\|_s^{p/2s}\big).
\end{aligned}$$

where Lemma 3.1 has been used, and C denotes constants which are independent of $f \in Y_s$. But since $Q(f_n) = q$ for all n, then $|f_n|_2$ remains bounded and so we have

$$\|f_n\|_s^2 \leq C\left(1 + \|f_n\|_s^{p/2s}\right).$$

Since $p/2s < 2$, the existence of the bound B follows immediately.

To prove (ii), suppose that such a constant δ_2 does not exist. Then

$$\liminf_{n\to\infty} \int_{-\infty}^{\infty} N(f_n)\, dx \leq 0.$$

Also, from the definition of Λ it follows that

$$\int_{-\infty}^{\infty} \langle f_n(x), \Lambda f_n(x)\rangle\; dx \geq 0$$

for all n. Hence

$$\begin{aligned} I_q &= \lim_{n\to\infty} E_0(f_n) \\ &= \lim_{n\to\infty} \left(\int_{-\infty}^{\infty} \frac{1}{2}\langle f_n(x), \Lambda f_n(x)\rangle\; dx - \int_{-\infty}^{\infty} N(f_n)\, dx\right) \\ &\geq -\liminf_{n\to\infty} \int_{-\infty}^{\infty} N(f_n)\, dx \geq 0, \end{aligned}$$

which contradicts the assumption that $I_q < 0$. □

Lemma 3.4. *For all* q_1, $q_2 > 0$, *one has*

$$I_{q_1+q_2} < I_{q_1} + I_{q_2}.$$

Proof. First we claim that for $\theta > 1$ and $q > 0$,

$$I_{\theta q} < \theta\, I_q.$$

In fact, let $\{f_n\}$ be a minimizing sequence for I_q, and notice that for all n, $Q(\sqrt{\theta} f_n) = \theta q$ and hence $E_0(\sqrt{\theta} f_n) \geq I_{\theta q}$. It follows that

$$I_{\theta q} \leq E_0(\sqrt{\theta} f_n) = \theta E_0(f_n) + (\theta - \theta^{(p+2)/2}) \int_{-\infty}^{\infty} N(f_n)\, dx;$$

and taking $n \to \infty$ and using Lemma 3.3(ii), we conclude that

$$I_{\theta q} \leq \theta I_q + (\theta - \theta^{(p+2)/2})\delta_2 < \theta I_q,$$

as claimed.

Now in case $q_1 > q_2$, then from what was just shown it follows that

$$\begin{aligned} I_{(q_1+q_2)} &= I_{q_1(1+q_2/q_1)} < (1 + \frac{q_2}{q_1})I_{q_1} \\ &< I_{q_1} + \frac{q_2}{q_1}\left(\frac{q_1}{q_2}I_{q_2}\right) = I_{q_1} + I_{q_2}; \end{aligned}$$

whereas in the case $q_1 = q_2$ we have

$$I_{(q_1+q_2)} = I_{2q_1} < 2I_{q_1} = I_{q_1} + I_{q_2}. \quad \square$$

The next step in the proof of Theorem 2.1 is to rule out the possibilities that $0 < \alpha < q$ and that $\alpha = 0$. The former of these two possibilities is dealt with in the next three lemmas, which represent a simplification and generalization of an argument appearing in Section 4 of [2].

Lemma 3.5. *Let*

$$P = \left[\frac{s}{2}\right] + 1,$$

where the brackets denote the greatest integer function. We can write $\Lambda = \Lambda_1 + (\Lambda_2)^2$, *where* Λ_1 *and* Λ_2 *are self-adjoint operators on* Y_s *with the following properties:*

(i) There exists a constant $C > 0$ *such that if* ζ *is any function which is in* $L^\infty(\mathbb{R})$ *and has derivative* ζ' *in* $L^\infty(\mathbb{R})$, *and* f *is any function in* X_2, *then*

$$|[\Lambda_1, \zeta]f|_2 \leq C|\zeta'|_\infty |f|_2,$$

where $[\Lambda_1, \zeta]$ *denotes the commutator* $\Lambda_1(\zeta f) - \zeta(\Lambda_1 f)$.

(ii) There exists a constant $C > 0$ *such that if* ζ *is any function which is in* $L^\infty(\mathbb{R})$ *and has derivatives up to order* P *in* $L^\infty(\mathbb{R})$, *and* f *is any function in* X_2, *then*

$$|[\Lambda_2, \zeta]f|_2 \leq C\left(\sum_{i=1}^{P}\left|\frac{d^i\zeta}{dx^i}\right|_\infty\right)|f|_2.$$

Proof. First choose a function $\chi(k) \in C_0^\infty(\mathbb{R})$ such that $\chi(k) = 1$ for $|k| < K$, where K is the constant defined in assumption (A2) above. Define $A_1(k) = \chi(k)(\sigma_0 D + A(k))$, and define $A_2(k)$ to be the square root of the positive definite matrix $(1 - \chi(k))(\sigma_0 D + A(k))$. Since $\sigma_0 D + A(k) = A_1(k) + (A_2(k))^2$, then $\Lambda = \Lambda_1 + \Lambda_2^2$, where Λ_1 and Λ_2 are the Fourier multiplier operators with symbols $A_1(k)$ and $A_2(k)$.

Now, for given values of i and j between 1 and n, let $(a_1)_{ij}(k)$ be the entry in the ith row and jth column of $A_1(k)$, and let $(\Lambda_1)_{ij}$ denote the scalar Fourier multiplier operator with symbol $(a_1)_{ij}(k)$. Let $\tilde{\Lambda} = (\Lambda_1)_{ij} - (a_1)_{ij}(0)$; then we can write $\tilde{\Lambda} = \frac{d}{dx} T$ where T is the operator with symbol

$$\sigma(k) = \frac{(a_1)_{ij}(k) - (a_1)_{ij}(0)}{k}.$$

By assumption (A2), we have that $\sup_{k \in \mathbb{R}} |k|^m \left| \left(\frac{d}{dk}\right)^m \sigma(k) \right| < \infty$ for $0 \le m \le 4$; and it then follows from Theorem 35 of [15] that

$$|[T, \zeta] f'|_2 \le C |\zeta'|_\infty |f|_2,$$

for some C independent of $f \in L^2(\mathbb{R})$ and ζ (As stated in [15], Theorem 35 actually requires estimates on σ for all $m \ge 0$, but the proof given there shows that it suffices to have estimates for $0 \le m \le 4$.) Since

$$|[(\Lambda_1)_{ij}, \zeta] f|_2 = \left| [\tilde{\Lambda}, \zeta] f \right|_2 = \left| T \frac{d}{dx}(\zeta f) - \zeta T f' \right|_2 \le |T(\zeta' f)|_2 + |[T, \zeta] f'|_2,$$

and T is bounded on L_2, it follows that

$$|[(\Lambda_1)_{ij}, \zeta] f|_2 \le C |\zeta'|_\infty |f|_2$$

for all $f \in L^2$. Finally, since for $f = (f_1, \ldots, f_n) \in X_2$ one has

$$|[\Lambda_1, \zeta] f|_2 = \left| \sum_{i,j=1}^n [(\Lambda_1)_{ij}, \zeta] f_j \right|_2 \le \sum_{i,j=1}^n |[(\Lambda_1)_{ij}, \zeta] f_j|_2 \,,$$

it follows that (i) holds for Λ_1.

Similarly, to prove (ii) it suffices to verify that the same statement holds for all $f \in L^2(\mathbb{R})$ if Λ_2 is replaced by its ijth entry $(\Lambda_2)_{ij}$. But this is exactly the content of part 2 of Lemma 4.2 of [2], since $(\Lambda_2)_{ij}$ has the same properties as the operator M_2 defined there. □

Lemma 3.6. *For every $\epsilon > 0$, there exist a number $N \in \mathbb{N}$ and sequences $\{g_N, g_{N+1}, \cdots\}$ and $\{h_N, h_{N+1}, \cdots\}$ of functions in Y_s such that for every $n \geq N$,*

(i) $|Q(g_n) - \alpha| < \epsilon$,

(ii) $|Q(h_n) - (q - \alpha)| < \epsilon$, *and*

(iii) $E_0(f_n) \geq E_0(g_n) + E_0(h_n) - \epsilon$.

Proof. Choose $\phi \in C_0^\infty$ with support in $[-2, 2]$ such that $\phi \equiv 1$ on $[-1, 1]$, and let $\psi \in C^\infty$ be such that $\phi^2 + \psi^2 \equiv 1$ on $\mathbb{R}$. For each $r \in \mathbb{R}$ define $\phi_r(x) = \phi(x/r)$ and $\psi_r(x) = \psi(x/r)$.

From the definition of α it follows that for every sufficiently large value of r, one can find $N = N(r)$ such that for all $n \geq N$,

$$\alpha - \epsilon < M_n(r) \leq M_n(2r) < \alpha + \epsilon.$$

In particular, we can find y_n such that

$$\int_{y_n - r}^{y_n + r} \frac{1}{2} \langle f, Df \rangle \, dx > \alpha - \epsilon$$

and

$$\int_{y_n - 2r}^{y_n + 2r} \frac{1}{2} \langle f, Df \rangle \, dx < \alpha + \epsilon.$$

It follows that if we define $g_n(x) = \phi_r(x - y_n) f_n(x)$ and $h_n(x) = \psi_r(x - y_n) f_n(x)$, then (i) and (ii) hold for all $n \geq N(r)$. We now show that if r is chosen sufficiently large, then (iii) also holds for all such n, if ϵ in (iii) is replaced by $C\epsilon^\mu$ for certain positive numbers C and μ.

Begin by writing

$$E_0(g_n) = \frac{1}{2} \left[\int_{-\infty}^{\infty} \langle g_n, \Lambda_1 g_n \rangle \, dx + \int_{-\infty}^{\infty} \langle \Lambda_2 g_n, \Lambda_2 g_n \rangle \, dx \right] - \int_{-\infty}^{\infty} N(g_n) \, dx. \tag{3.1}$$

The first of the integrals on the right-hand side of (3.1) can be written as

$$\int_{-\infty}^{\infty} \langle \phi_r f_n, \Lambda_1(\phi_r f_n) \rangle \, dx = \int_{-\infty}^{\infty} \phi_r^2 \langle f_n, \Lambda_1 f_n \rangle \, dx + \int_{-\infty}^{\infty} \langle \phi_r f_n, [\Lambda_1, \phi_r] f_n \rangle \, dx.$$

Now by Lemma 3.5 (i),

$$\begin{aligned} \left| \int_{-\infty}^{\infty} \langle \phi_r f_n, [\Lambda_1, \phi_r] f_n \rangle \, dx \right| &\leq |\phi_r f_n|_2 \, |[\Lambda_1, \phi_r] f_n|_2 \\ &\leq C |\phi_r'|_\infty |f_n|_2^2. \end{aligned}$$

But since $|\phi_r'|_\infty = |\phi'|_\infty/r$, and $|f_n|_2$ is bounded independently of n, it follows that

$$\left|\int_{-\infty}^{\infty} \langle \phi_r f_n, [L, \phi_r] f_n\rangle \, dx\right| \leq C/r$$

where the constant C is independent of r, n and ϵ.

Similarly, writing the second integral on the right-hand side of (3.1) as

$$\int_{-\infty}^{\infty} \phi_r^2 \langle \Lambda_2 f_n, \Lambda_2 f_n\rangle \, dx + 2\int_{-\infty}^{\infty} \phi_r \langle \Lambda_2 f_n, [\Lambda_2, \phi_r] f_n\rangle \, dx + |[\Lambda_2, \phi_r] f_n|_2^2 \,,$$

and using Lemma 3.5 (ii) and the fact that Λ_2 is a bounded operator from Y_s to X_2, we see that

$$\int_{-\infty}^{\infty} \langle \Lambda_2 g_n, \Lambda_2 g_n\rangle \, dx \leq \int_{-\infty}^{\infty} \phi_r^2 \langle \Lambda_2 f_n, \Lambda_2 f_n\rangle \, dx + C/r,$$

where again C is independent of r, n and ϵ.

Finally, since

$$\begin{aligned}\left|\int_{-\infty}^{\infty} N(g_n) - \phi_r^2 \, N(f_n) \ dx\right| &= \left|\int_{-\infty}^{\infty} (\phi_r^{p+2} - \phi_r^2) N(f_n) \, dx\right| \\ &\leq C(|f_n|_{X_{p+2}(I_1)} + |f_n|_{X_{p+2}(I_1)})^{p+2},\end{aligned}$$

where I_1 and I_2 denote the intervals $[y_n - 2r, y_n - r]$ and $[y_n + r, y_n + 2r]$, it follows from Lemma 3.1 that

$$\begin{aligned}\left|\int_{-\infty}^{\infty} N(g_n) - \phi_r^2 \, N(f_n) \ dx\right| &\leq C\|f_n\|_s^{p/2s} \left(|f_n|_{X_2(I_1)} + |f_n|_{X_2(I_2)}\right)^{p+2-(p/2s)} \\ &\leq C\epsilon^\mu,\end{aligned}$$

where $\mu = p + 2 - (p/2s)$ and C is independent of r, n, and ϵ.

Substituting these inequalities in (3.1) yields

$$E_0(g_n) \leq \int_{-\infty}^{\infty} \phi_r^2 \left(\frac{1}{2}\langle f_n, \Lambda f_n\rangle - N(f_n)\right) \ dx + C(1/r + \epsilon^\mu).$$

The same argument yields the result

$$E_0(h_n) \leq \int_{-\infty}^{\infty} \psi_r^2 \left(\frac{1}{2}\langle f_n, \Lambda f_n\rangle - N(f_n)\right) \ dx + C(1/r + \epsilon^\mu),$$

and it follows that

$$E_0(g_n) + E_0(h_n) \leq E(f_n) + C(1/r + \epsilon^\mu).$$

Choosing $r \geq 1/\epsilon^{\mu}$, we conclude that there exists a constant C, independent of ϵ, such that

$$E(f_n) \geq E(g_n) + E(h_n) - C\epsilon^{\mu}$$

for all $n \geq N(r)$.

This proves the Lemma, except that (iii) has been modified by replacing ϵ by $C\epsilon^{\mu}$. But since C and μ are independent of ϵ, we can now apply what has just been proved to $\tilde{\epsilon}$, where $\tilde{\epsilon}$ is chosen to be less than the minimum of ϵ and $(\epsilon/C)^{1/\mu}$; it follows that the Lemma holds as stated. □

Lemma 3.7. *If* $0 < \alpha < q$ *then*

$$I_q \geq I_\alpha + I_{q-\alpha}.$$

Proof. First, we claim that if γ is any real number and $f \in Y$ with $\|f\|_Y \leq B$ and $|Q(f) - \gamma| \leq \gamma/2$, then

$$I_\gamma \leq E_0(f) + C|Q(f) - \gamma|,$$

where C depends only on γ and B. To see this, let $\tilde{f} = \sqrt{\theta} f$ where $\theta = \gamma/Q(f)$. Then $Q(\tilde{f}) = \gamma$, and so

$$\begin{aligned} I_\gamma \leq E_0(\tilde{f}) &= E_0(f) + (\theta - 1)E_0(f) + \theta(1 - \theta^{p/2}) \int_{-\infty}^{\infty} N(f)\, dx \\ &\leq E_0(f) + C\left(|1 - \theta| + \theta|1 - \theta^{p/2}|\right). \end{aligned}$$

But $|Q(f) - \gamma| \leq \gamma/2$ implies that $\theta \leq 2$ and that $|1 - \theta^{p/2}| < C|1 - \theta| < C|Q(f) - \gamma|$, so the claim has been proved.

The preceding observation together with Lemma 3.6 implies that there exists a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and corresponding functions $\{g_{n_k}\}$ and $\{h_{n_k}\}$ such that for all k,

$$\begin{aligned} &E_0(g_{n_k}) \geq I_\alpha - \frac{1}{k} \\ &E_0(h_{n_k}) \geq I_{q-\alpha} - \frac{1}{k} \\ &E_0(f_{n_k}) \geq E_0(g_{n_k}) + E_0(h_{n_k}) - \frac{1}{k}. \end{aligned}$$

Thus

$$E_0(f_{n_k}) \geq I_\alpha + I_{q-\alpha} - \frac{3}{k},$$

and the desired result follows by taking the limit of both sides as $k \to \infty$. $\square$

The next two lemmas are used to dispose of the possibility that $\alpha = 0$.

Lemma 3.8. *Suppose $B > 0$ and $\delta > 0$ are given. Then there exists $\delta_1 = \delta_1(B, \delta) > 0$ such that if $f \in Y_s$ with $\|f\|_s \leq B$ and $|f|_{p+2} \geq \delta$, then*

$$|f|_{X_{p+2}(I)} \geq \delta_1$$

for some interval $I \subset \mathbb{R}$ of length 4.

Proof. Choose $\chi : \mathbb{R} \to [0,1]$ smooth with support in $[-2,2]$ and satisfying $\sum_{j\in\mathbb{Z}} \chi(x-j) = 1$ for all $x \in \mathbb{R}$, and define $\chi_j = \chi(x-j)$ for all $j \in \mathbb{Z}$. The map $T : Y_s \to l_2(Y_s)$ defined by

$$Tw = \{\chi_j\, w\}_{j\in\mathbb{Z}}$$

is bounded (this is clear in the case when s is a non-negative integer, and the case for general $s \geq 0$ then follows by interpolation: see, e.g., Section 5.6 of [8]). Therefore we can find $C_0 > 0$ such that

$$\sum_{j\in\mathbb{Z}} \|\chi_j\, f\|_s^2 \leq C_0 \|f\|_s^2$$

for all $f \in Y_s$.

Now let C_1 be a positive number such that $\sum_{j\in\mathbb{Z}} |\chi(x-j)|^3 \geq C_1$ for all $x \in \mathbb{R}$, and define $C_2 = \frac{C_0 B^2}{C_1}$. We claim that for every nonzero $f \in Y_s$ there exists $j_0 \in \mathbb{Z}$ such that

$$\|\chi_{j_0} f\|_s^2 \leq (1 + C_2 |f|_{p+2}^{-(p+2)}) |\chi_{j_0} f|_{p+2}^{p+2}.$$

In fact, if no such j_0 exists, then one has

$$\|\chi_j f\|_s^2 > (1 + C_2 |f|_{p+2}^{-(p+2)}) |\chi_j f|_{p+2}^{p+2}$$

for every $j \in \mathbb{Z}$. But then summing over j leads to

$$C_0 B^2 > (1 + C_2 |f|_{p+2}^{-(p+2)}) C_1 |f|_{p+2}^{p+2} = C_1 |f|_{p+2}^{p+2} + C_0 B^2,$$

which is a contradiction.

Since $|f|_{p+2} \geq \delta$, it follows from our claim that

$$\|\chi_{j_0} f\|_s^2 \leq (1 + C_2\delta^{-(p+2)})|\chi_{j_0} f|_{p+2}^{p+2}.$$

Now since $s > p/4 > p/(2p+4)$, from Sobolev's embedding theorem it follows that

$$|\chi_{j_0} f|_{p+2} \leq |f|_{p+2} \leq C_3\|f\|_s,$$

with C_3 independent of f. Therefore

$$|\chi_{j_0} f|_{p+2} \geq (C_3^2(1 + C_2\delta^{-(p+2)}))^{-1/p},$$

and hence the Lemma has been proved, with $\delta_1 = (C_3^2(1 + C_2\delta^{-(p+2)}))^{-1/p}$ and $I = [j_0 - 2, j_0 + 2]$. $\square$

Lemma 3.9. *For every minimizing sequence $\{f_n\}$, we have $\alpha > 0$.*

Proof. From Lemmas 3.3 and 3.8 we deduce that there exist $\delta_1 > 0$ and a sequence of intervals $\{I_n\} = \{[y_n - 2, y_n + 2]\}$ such that

$$|f_n|_{X_{p+2}(I_n)}^{p+2} \geq \delta_1$$

for all sufficiently large n. Then Lemma 3.1, together with Lemma 3.3(i), gives

$$\delta_1 \leq CB^{p/2s}\,|f_n|_{X_2(I_n)}^{\mu} \leq C\left(\int_{y_n-2}^{y_n+2} \langle f_n, Df_n\rangle\, dx\right)^{\mu/2}$$

for all sufficiently large n, where $\mu = p + 2 - (p/2s)$ and C is independent of n. Hence

$$\alpha = \lim_{r\to\infty} M(r) \geq M(2) = \lim_{n\to\infty} M_n(2) \geq \frac{1}{2}\left(\frac{\delta_1}{C}\right)^{2/\mu} > 0. \quad \square$$

Note now that Lemmas 3.4, 3.7, and 3.9 combine to show that $\alpha = q$. Therefore we can apply the following result:

Lemma 3.10. *Suppose $\alpha = q$. Then there exists a sequence of real numbers $\{y_1, y_2, \cdots\}$ such that the sequence $\{\tilde{f}_n\}$ defined by*

$$\tilde{f}_n(x) = f_n(x + y_n) \text{ for all } x \in \mathbb{R}$$

has a subsequence converging in Y_s norm to a function $g \in G_q$.

We omit the proof of Lemma 3.10, since it differs in only minor details from the proof of Lemma 2.5 of [2]; and the modifications which are required are obvious.

Lemma 3.11. *The set G_q is not empty. Moreover, if $\{f_n\}$ is any minimizing sequence for I_q, then*

(i) there exists a sequence $\{y_1, y_2, ...\}$ and an element $g \in G_q$ such that $f_n(\cdot + y_n)$ has a subsequence converging strongly in Y_s to g.

(ii)

$$\lim_{n\to\infty} \inf_{\substack{g\in G_q\\ y\in\mathbb{R}}} \|f_n(\cdot + y) - g\|_s = 0.$$

(iii) f_n converges to G_q in Y_s.

The same conclusions hold for $\{f_n\}$ under the weaker hypothesis that $Q(f_n) \to q$ and $E_0(f_n) \to I_q$ as $n \to \infty$.

Proof. Lemmas 3.4, 3.7 and 3.9 show that $\alpha = q$; it then follows from Lemma 3.10 that G_q is nonempty and that (i) holds for any minimizing sequence $\{f_n\}$. If, on the other hand, we assume only that $Q(f_n) \to q$ as $n \to \infty$, then we still can assert that (i) holds for the minimizing sequence $\alpha_n f_n$, where $\alpha_n = \sqrt{q/Q(f_n)}$. But since $\alpha_n \to 1$, the convergence of a subsequence of $\alpha_n f_n(\cdot + y_n)$ to g in Y_s implies the convergence of the same subsequence of $f_n(\cdot + y_n)$ to g. Thus (i) holds under the weaker hypothesis on $\{f_n\}$.

To complete the proof it suffices to show that (i) implies (ii) and (iii). But (ii) follows immediately from (i) and the fact that every subsequence of a minimizing sequence is itself a minimizing sequence; and (iii) follows from (ii) and the fact that the functionals E_0 and Q (and hence also the set G_q) are invariant under the operation of replacing f by $f(\cdot + y)$. $\square$

We can now complete the proof of Theorem 2.1. It has already been shown in Lemma 3.11 that G_q is non-empty. It remains therefore to show that the solitary waves in G_q have wavespeeds greater than σ_0, and that the set G_q is stable.

It follows from the definition of G_q and the Lagrange multiplier principle (cf. Theorem 7.7.2 of [28]) that for each $g \in G_q$ there exists $\lambda \in \mathbb{R}$ such that

$$\delta E_0(g) = \lambda\, \delta Q(g),$$

where the Fréchet derivatives $\delta E_0(g)$ and $\delta Q(g)$ are given by

$$\begin{aligned}\delta E_0(g) &= \Lambda g - \nabla N(g) = \sigma_0 Dg + Lg - \nabla N(g),\\ \delta Q(g) &= Dg.\end{aligned}$$

Hence g solves (2.2) with $c = \sigma_0 - \lambda$; i.e., the wavespeed of the solitary wave g is $\sigma_0 - \lambda$. We wish to show that $\lambda < 0$.

Note first that

$$\begin{aligned}\frac{d}{d\theta}\left[E_0(\theta g)\right]_{\theta=1} &= \frac{d}{d\theta}\left[\theta^2 \int_{-\infty}^{\infty} \frac{1}{2}\langle g, \Lambda g\rangle \, dx - \theta^{p+2}\int_{-\infty}^{\infty} N(g)\, dx\right]_{\theta=1}\\ &= \int_{-\infty}^{\infty} \langle g, \Lambda g\rangle\, dx - (p+2)\int_{-\infty}^{\infty} N(g)\, dx\\ &= 2E_0(g) - p\int_{-\infty}^{\infty} N(g)\, dx.\end{aligned}$$

But $E_0(g) = I_q < 0$, and $\int_{-\infty}^{\infty} N(g)\, dx > 0$ by Lemma 3.3(ii), so

$$\frac{d}{d\theta}\left[E_0(\theta g)\right]_{\theta=1} < 0.$$

On the other hand, from the definition of the Fréchet derivative we have

$$\begin{aligned}\frac{d}{d\theta}\left[E_0(\theta g)\right]_{\theta=1} &= \int_{-\infty}^{\infty} \langle \delta E_0(g), \frac{d}{d\theta}\left[\theta g\right]_{\theta=1}\rangle\, dx\\ &= \lambda \int_{-\infty}^{\infty} \langle \delta Q(g), g\rangle\, dx = \lambda\int_{-\infty}^{\infty}\langle g, Dg\rangle\, dx;\end{aligned}$$

and since $\int_{-\infty}^{\infty} \langle g, Dg\rangle\, dx > 0$ it follows that $\lambda < 0$ as claimed.

Now suppose that G_q is not stable. Then there exists a sequence of solutions $\{h_n\}$ of (2.1) and a sequence of times $\{t_n\}$ such that $h_n(\cdot, 0)$ converges to G_q in Y_s, but $h_n(\cdot, t_n)$ does not converge to G_q in Y_s. Since E_0 and Q are constants of the motion for (2.1) and are continuous on Y_s, it follows that $Q(h_n(\cdot, t_n)) \to q$ and $E_0(h_n(\cdot, t_n)) \to I_q$ as $n \to \infty$. Hence from Lemma 9(iii) it follows that $h_n(\cdot, t_n)$ converges to G_q in Y_s, a contradiction. $\square$

4. Applications to model systems for long waves.

a) The Gear-Grimshaw system.

The Gear-Grimshaw system was derived in [20] to model the strong interaction of two long internal gravity waves in a stratified fluid, where the two waves are assumed to correspond to different modes of the linearized equations of motion. Following [11], we write it as

$$\begin{aligned} h_{1t} + h_1 h_{1x} + a_1 h_2 h_{2x} + a_2 (h_1 h_2)_x + h_{1xxx} + a_3 h_{2xxx} &= 0 \\ b_1 h_{2t} + r h_{2x} + h_2 h_{2x} + b_2 a_2 h_1 h_{1x} + b_2 a_1 (h_1 h_2)_x + b_2 a_3 h_{1xxx} + h_{2xxx} &= 0, \end{aligned} \tag{4.1}$$

where a_1, a_2, a_3, b_1, b_2, and r are real constants with b_1, b_2 positive.

The system (4.1) can be rewritten in the form (2.1) by putting

$$N(h_1, h_2) = \frac{1}{2}\left(\frac{h_1^3}{3} + a_1 h_1 h_2^2 + a_2 h_1^2 h_2 + \frac{b_2^{-1} h_2^3}{3}\right)$$

and

$$D = \begin{bmatrix} 1 & 0 \\ 0 & b_1 b_2^{-1} \end{bmatrix};$$

and defining the symbol $A(k)$ of L by

$$A(k) = \begin{bmatrix} k^2 & a_3 k^2 \\ a_3 k^2 & b_2^{-1} k^2 \end{bmatrix} = k^2 \begin{bmatrix} 1 & a_3 \\ a_3 & b_2^{-1} \end{bmatrix}.$$

We verify that (4.1) satisfies the assumptions required by the stability theory of Section 2. First, in [11], it was shown that if $b_2 a_3^2 < 1$, then (4.1) is globally well-posed in Y_r for every $r \geq 1$. In the case of (4.1), the function N appearing in (2.1) is homogeneous of degree 3, so we take $p = 1$. The signs of the eigenvalues of $A(k)$ are independent of k, and both are positive if and only if $b_2 a_3^2 < 1$, so (A3) holds in this case with $s = 1$. One sees easily that then $\sigma_0 = 0$. From the formula for N, we see that no matter what the values of the parameters a_i and b_i, one can always find $v_0 \in \mathbb{R}^2$ such that $N(v_0) > 0$, and (2.4) obviously holds for any $v_0 \in \mathbb{R}^2$, with $s_0 = 2$. Hence from Theorem 2.2 we obtain the following result.

Theorem 4.1. *Suppose that $b_2 a_3^2 < 1$. Let E and Q be the invariant functionals associated with (4.1), as defined in Section 2. Then for each $q > 0$, the problem of minimizing E subject to the constraint $Q = q$ has a nonempty solution set G_q, and for each $g \in G_q$, there exists $c > 0$ such that $g(x - ct)$ is a solution of (4.1). Moreover, the set G_q is stable in the sense that for every $\epsilon > 0$, there exists $\delta > 0$ with the following property: if h_0 is any function in Y_1 satisfying*

$$\|h_0 - g\|_1 < \delta$$

for some $g \in G_q$, then there exists a global solution $h(x,t)$ of (4.1) with $h(x,0) = h_0$ and a map $t \to g(t)$ from $[0,\infty)$ to G_q such that

$$\|h(\cdot,t) - g(t)\|_1 < \epsilon$$

for all $t \geq 0$.

b) The Liu-Kubota-Ko system.

The Liu, Kubota & Ko system was derived in [27] to model the interaction between a disturbance $h_1(x,t)$ located at an upper pycnocline and another disturbance $h_2(x,t)$ located at a lower pycnocline in a three-layer fluid. It can be written as

$$\begin{aligned} &h_{1t} - c_1 h_{1x} + \alpha_1 h_1 h_{1x} - \gamma_1 (M_1 h_1)_x - \gamma_2[(M_2 h_1)_x - (S h_2)_x] = 0 \\ &h_{2t} - c_2 h_{2x} + \alpha_2 h_2 h_{2x} - \gamma_3 (M_3 h_2)_x - \gamma_4[(M_2 h_2)_x - (S h_1)_x] = 0. \end{aligned} \tag{4.2}$$

Here c_1, c_2, α_1, α_2, γ_1, γ_2, γ_3, γ_4 are real constants, with γ_i positive for $i = 1, 2, 3, 4$. The operators M_1, M_2, M_3 are Fourier multiplier operators defined for $\zeta \in H^{1/2}$ by

$$\widehat{M_i \zeta}(k) = m_i(k)\widehat{\zeta}(k),$$

where

$$m_i(k) = k \coth(k H_i) - \frac{1}{H_i}$$

for $i = 1, 2, 3$; with H_1, H_2, H_3 being positive constants related to the depths of the three fluid layers. The operator S is also a Fourier multiplier operator,

$$\widehat{S\zeta}(k) = n(k)\widehat{\zeta}(k),$$

where

$$n(k) = \frac{k}{\sinh k H_2}\,.$$

The system (4.2) can be rewritten in the form of (2.1), with $n = 2$ and $h = (h_1, h_2)$, by putting

$$N(h) = \frac{1}{6}(\alpha_1 \gamma_4 h_1^3, \alpha_2 \gamma_2 h_2^3)$$

and

$$D = \begin{bmatrix} \gamma_4 & 0 \\ 0 & \gamma_2 \end{bmatrix},$$

and defining the symbol $A(k)$ of L by

$$A(k) = \begin{bmatrix} \gamma_4(c_1 + \gamma_1 m_1(k) + \gamma_2 m_2(k)) & -\gamma_2\gamma_4 n(k) \\ -\gamma_2\gamma_4 n(k) & \gamma_2(c_2 + \gamma_4 m_2(k) + \gamma_3 m_3(k)) \end{bmatrix}.$$

In Theorem 2.3 of [3] it is shown that (4.2) is globally well-posed in Y_r for any $r \geq 3/2$. As in the case of (4.1), the functional N is homogeneous of degree 3, so we take $p = 1$. The eigenvalues of $\sigma D + A(k)$ are given by (cf. [3])

$$\lambda_1(k, \sigma) = \frac{T(k)}{2} - \sqrt{T(k)^2 - 4d(k)}$$
$$\lambda_2(k, \sigma) = \frac{T(k)}{2} + \sqrt{T(k)^2 - 4d(k)},$$

where $T(k)$ is the trace of $\sigma D + A(k)$ and $d(k)$ is the determinant of $\sigma D + A(k)$. It follows easily from the properties of $m_i(k)$ and $n(k)$ that (A3) (and hence also (A1) and (A2)) is satisfied with $s = 1/2$, and that σ_0 is the larger of the two roots of the equation

$$(\sigma + c_1)(\sigma + c_2) = \frac{\gamma_2\gamma_4}{H_2^2};$$

i.e.

$$\sigma_0 = \frac{1}{2}\left[-(c_1 + c_2) + \sqrt{(c_1 - c_2)^2 + \frac{4\gamma_2\gamma_4}{H_2^2}}\right]. \tag{4.3}$$

Moreover, $\sigma_0 D + A(k)$ has the eigenvalue 0 at $k = 0$, so that (2.4) holds with $s_0 = 1$, by the first remark following Theorem 2.2. Also, $N(-h) = -N(h)$, so that the second remark following Theorem 2.2 applies. It follows that all the assumptions underlying Theorem 2.2 are satisfied, and we obtain the following result.

Theorem 4.2. *Let E and Q be the invariant functionals associated with (4.2), as defined in Section 2. Then for each $q > 0$, the problem of minimizing E subject to the constraint $Q = q$ has a nonempty solution set G_q, and for each $g \in G_q$, there exists $c > \sigma_0$ such that $g(x - ct)$ is a solution of (4.2). Moreover, the set G_q is stable in the sense that for every $\epsilon > 0$, there exists $\delta > 0$ with the following property: if h_0 is any function in $Y_{3/2}$ satisfying*

$$\|h_0 - g\|_{1/2} < \delta$$

for some $g \in G_q$*, then there exists a global solution* $h(x,t)$ *of (4.2) with* $h(x,0) = h_0$ *and a map* $t \to g(t)$ *from* $[0,\infty)$ *to* G_q *such that*

$$\|h(\cdot,t) - g(t)\|_{1/2} < \epsilon$$

for all $t \geq 0$.

5. Symmetry of Liu-Kubota-Ko solitary waves.

We begin this section with a lemma that establishes a correspondence between solitary-wave solutions of (4.2) and solutions of a certain nonlinear boundary-value problem for the Laplacian, posed on the three infinite strips S_1, S_2, S_3 defined as subsets of $\mathbb{R}^2$ by

$$S_1 = \mathbb{R} \times [0, H_1],$$
$$S_2 = \mathbb{R} \times [-H_2, 0],$$
$$S_3 = \mathbb{R} \times [-(H_2 + H_3), -H_2].$$

Lemma 5.1. *Let* $\phi = (\phi_1, \phi_2) \in X_2$ *be such that* $\phi(x - ct)$ *solves (4.2) for some* $c > \sigma_0$*, where* σ_0 *is as defined in (4.3). Then there exist functions* $u_i \in C^\infty(S_i)$*,* $i \in \{1,2,3\}$*, such that*

(i) *for* $i = 1,2,3$, $\Delta u_i = 0$ *on* S_i,

(ii) *for* $i = 1,2,3$, $u_i(x,y) \to 0$ *uniformly in* y *as* $|x| \to \infty$,

(iii) $u_1 = 0$ *for* $y = H_1$,

(iv) $u_3 = 0$ *for* $y = -(H_2 + H_3)$,

(v) $u_1 = u_2 = \phi_1$ *for* $y = 0$,

(vi) $u_2 = u_3 = \phi_2$ *for* $y = -H_2$,

(vii) $\left[-(c + c_1) + \frac{\gamma_1}{H_1} + \frac{\gamma_2}{H_2}\right] \phi_1 + \frac{\alpha_1}{2}\phi_1^2 + \gamma_1 u_{1y} - \gamma_2 u_{2y} = 0$ *for* $y = 0$,

(viii) $\left[-(c + c_2) + \frac{\gamma_3}{H_3} + \frac{\gamma_4}{H_2}\right] \phi_2 + \frac{\alpha_2}{2}\phi_2^2 + \gamma_4 u_{2y} - \gamma_3 u_{3y} = 0$ *for* $y = -H_2$.

Proof. As shown in Lemmas 4.2 and 4.3 of [3], if $\phi \in X_2$ is any solitary-wave solution of (4.2) with wavespeed $c > \sigma_0$, then ϕ must in fact be in Y_∞. Therefore, if we define u_1 on

S_1, u_2 on S_2, and u_3 on S_3 by the formulas

$$u_1(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\left(\frac{\sinh k(H_1-y)}{\sinh kH_1}\right)\widehat{\phi_1(k)}\;dk,$$
$$u_2(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\left[\left(\frac{\sinh k(H_2+y)}{\sinh kH_2}\right)\widehat{\phi_1(k)} - \left(\frac{\sinh ky}{\sinh kH_2}\right)\widehat{\phi_2(k)}\right]\;dk,$$
$$u_3(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\left(\frac{\sinh k(H_2+H_3+y)}{\sinh k(H_2+H_3)}\right)\widehat{\phi_2(k)}\;dk,$$

it follows from standard arguments (see the proof of Lemma 2 in [5]) that $u_i \in C^\infty(S_i)$ and tends to 0 uniformly in y as $|x| \to \infty$, and that the partial derivatives of u_i on S_i may be computed by differentiating under the integral. In particular, differentiation under the integral shows that (i) holds, and also that

$$u_{1y}\Big|_{y=0} = -M_1\phi_1 - \frac{1}{H_1}\phi_1,$$
$$u_{2y}\Big|_{y=0} = -S\phi_2 + M_2\phi_1 + \frac{1}{H_2}\phi_1,$$
$$u_{2y}\Big|_{y=-H_2} = S\phi_1 - M_2\phi_2 - \frac{1}{H_2}\phi_2,$$
$$u_{3y}\Big|_{y=-H_2} = M_3\phi_2 + \frac{1}{H_3}\phi_2.$$

Substitution of these expressions in the solitary-wave equation for ϕ yields (vii) and (viii). Finally, (iii)–(vi) are obvious from the definitions of the functions u_i. $\square$

Remark. The converse of Lemma 5.1 holds, in the following sense. For arbitrary $\phi \in Y_1$, there are unique harmonic functions u_i defined on the interior of S_i such that $|u_i(\cdot,y)|_{L^2}$ is uniformly bounded in y, and

$$\lim_{y\downarrow 0} u_1 = \lim_{y\uparrow 0} u_2 = \phi_1,$$
$$\lim_{y\downarrow -H_2} u_2 = \lim_{y\uparrow -H_2} u_3 = \phi_2,$$

where the limits are taken in the L^2 sense. The derivatives u_{iy} are also well-defined as L^2 traces on $\{y=0\}$ and $\{y=-H_2\}$. If (vii) and (viii) hold, then $\phi(x-ct)$ is a solution of (4.2).

We will work below not with the functions u_i themselves, but instead with functions $\bar{u}_i$ which we now proceed to define. The assumption $c > \sigma_0$ implies that $c + c_1$ and $c + c_2$ are positive and satisfy

$$(c+c_1)(c+c_2) > \frac{\gamma_2\gamma_4}{H_2^2}.$$

Therefore it is possible to find a number θ_2 such that $1+\theta_2 H_2$ is positive and satisfies

$$\frac{\gamma_4}{(c+c_2)H_2} < 1+\theta_2 H_2 < \frac{(c+c_1)H_2}{\gamma_2}.$$

Once θ_2 has been chosen, we can choose θ_1 and θ_3 such that $1+\theta_1 H_1$ and $1+\theta_2 H_2+\theta_3 H_3$ are positive and satisfy

$$(1+\theta_1 H_1) < \frac{\gamma_2 H_1}{\gamma_1 H_2}\left[\frac{(c+c_1)H_2}{\gamma_2} - (1+\theta_2 H_2)\right] \tag{5.1}$$

and

$$(1+\theta_2 H_2+\theta_3 H_3) < \frac{(c+c_2)H_3}{\gamma_3}\left[(1+\theta_2 H_2) - \frac{\gamma_4}{(c+c_2)H_2}\right]. \tag{5.2}$$

Define

$$\begin{aligned} g_1(y) &= 1+\theta_1 y && \text{for } 0 \le y \le H_1, \\ g_2(y) &= 1-\theta_2 y, && \text{for } -H_2 \le y \le 0, \\ g_3(y) &= 1+\theta_2 H_2 - \theta_3(y+H_2) && \text{for } -(H_2+H_3) \le y \le -H_2. \end{aligned}$$

Notice that $g_1(0) = g_2(0)$ and $g_2(-H_2) = g_3(-H_2)$, and that each function $g_i(y)$ takes only positive values on its domain. Hence we may define functions $\bar{u}_i$ on S_i for $i = 1,2,3$ by

$$u_i(x,y) = g_i(y)\bar{u}_i(x,y).$$

Properties (ii), (iii), and (iv) of Lemma 5.1 still hold with u_i replaced by $\bar{u}_i$, and it is still true, as in (v) and (vi), that $\bar{u}_1 = \bar{u}_2$ for $y = 0$ and $\bar{u}_2 = \bar{u}_3$ for $y = -H_2$. Also, although the functions $\bar{u}_i$ are no longer harmonic, they do satisfy the elliptic equation

$$\Delta \bar{u}_i + \frac{2g_i'(y)}{g_i(y)}\bar{u}_{iy} = 0 \tag{5.3}$$

on S_i. Finally, we see from equations (vii) and (viii) that

$$Q_1\bar{u}_2 + \frac{\alpha_1}{2}\bar{u}_2^2 + \gamma_1\bar{u}_{1y} - \gamma_2\bar{u}_{2y} = 0 \qquad \text{for } y = 0 \tag{5.4}$$

and

$$Q_2\bar{u}_2 + \frac{\alpha_2}{2}\bar{u}_2^2 + \gamma_4(1+\theta_2 H_2)\bar{u}_{2y} - \gamma_3(1+\theta_2 H_2)\bar{u}_{3y} = 0 \qquad \text{for } y = -H_2, \tag{5.5}$$

where

$$Q_1 = -(c+c_1) + \frac{\gamma_1}{H_1}(1+\theta_1 H_1) + \frac{\gamma_2}{H_2}(1+\theta_2 H_2),$$
$$Q_2 = -(c+c_2)(1+\theta_2 H_2) + \frac{\gamma_4}{H_2} + \frac{\gamma_3}{H_3}(1+\theta_2 H_2 + \theta_3 H_3).$$

From (5.1) and (5.2), we see that both Q_1 and Q_2 are negative. Notice that such was not necessarily the case for the coefficients of ϕ_1 and ϕ_2 in (vii) and (viii) of Lemma 5.1. This is the reason for working with $\bar{u}_i$ instead of u_i.

In what follows, we make repeated use of the fact that solutions of (5.3) satisfy a maximum principle: if Ω is a bounded connected domain in $\mathbb{R}^2$, and $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies (5.3) on Ω, then u must achieve its maximum and minimum values over $\bar{\Omega}$ on the boundary of Ω. Further, if either of these values is attained at any point within Ω, then u is constant on Ω (see Theorem 3.5 of [22]). On an unbounded domain such as S_i, similar assertions can be made in the presence of additional assumptions on the behavior of u at infinity. For example, suppose u satisfies (5.3) on S_i and $u \to 0$, uniformly in y, as $|x| \to \infty$. By applying the maximum principle on sets $S_i \cap \{-R \le x \le R\}$ as $R \to \infty$, we can deduce that if u takes a negative value anywhere on S_i, then the minimum value of u over S_i must be attained at some point on the boundary of S_i.

We will also use the following refinements of the maximum principle, which are valid on any domain $\Omega \subset \mathbb{R}^2$, bounded or unbounded. The *Hopf boundary lemma* implies that if u satisfies (5.3) on Ω and attains its minimum value over $\bar{\Omega}$ at a point (x_0, y_0) on the boundary of Ω, and there exists a ball in Ω whose boundary contains (x_0, y_0), then the normal derivative of u at (x_0, y_0) is zero only if u is constant on $\bar{\Omega}$ (see Lemma 3.4 of [22]). There is also a *Hopf corner-point lemma* [21], which has the following implication for (5.3). Suppose $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ satisfies (5.3) on Ω, where Ω is the semi-infinite strip $(-\infty, x_0) \times (y_0, y_1)$, and u is non-negative on Ω and tends to 0, uniformly in y as $x \to -\infty$. Let P be a corner point of Ω; i.e., $P = (x_0, y_0)$ or $P = (x_0, y_1)$. If $u = 0$ at P, and the (one-sided) derivatives u_x, u_y, u_{xx}, u_{xy}, and u_{yy} exist and are all equal to 0 at P, then u is identically zero on Ω.

Lemma 5.2. *Suppose α_1 and α_2 are positive. Then $\bar{u}_i \geq 0$ on S_i for $i = 1, 2, 3$.*

Proof. Suppose, to the contrary, that for some i, $\bar{u}_i$ takes a negative value at some point of S_i. We will show that then either (5.4) or (5.5) must fail to hold. Since $\bar{u}_i$ satisfies (5.3) on S_i and tends to 0 uniformly in y as $|x| \to \infty$, it follows from the maximum principle that the minimum value of $\bar{u}_i$ on S_i must be attained at some point on the boundary of S_i. Furthermore, since $\bar{u}_1 = 0$ for $y = H_1$ and $\bar{u}_3 = 0$ for $y = -(H_2 + H_3)$, this negative minimum can only be attained on the boundary of S_2, where $\bar{u}_i = \bar{u}_2$. Hence $\bar{u}_2$ must take a negative minimum value at some point (x_0, y_0) on the boundary of S_2.

There are now two possibilities: either $y_0 = 0$ or $y_0 = -H_2$. In the first case, $(x_0, 0)$ is a minimum value both for $\bar{u}_1$ on S_1 and for $\bar{u}_2$ on S_2, so we must have that $\bar{u}_{1y}(x_0, 0) \geq 0$ and $\bar{u}_{2y}(x_0, 0) \leq 0$. On the other hand, $Q_1 < 0$ and $\bar{u}_2(x_0, 0) < 0$. Combining these facts, we conclude that the left-hand side of equation (5.4) is strictly positive, so (5.4) is contradicted. An exactly similar argument shows that if $y_0 = -H_2$, then (5.5) is contradicted. Thus the proof is complete. $\square$

Remark. From Theorem 5.4(ii) below it follows that if α_1 and α_2 are positive and the functions $\bar{u}_i$ are not all identically zero, then $\phi_1(x)$ and $\phi_2(x)$ are in fact strictly positive functions of x (and hence the $\bar{u}_i$ are strictly positive at all points in their domains except where $y = H_1$ and $y = -(H_2 + H_3)$). This fact is also a direct consequence of the proof of Lemma 5.1, since if ϕ_1 or ϕ_2 vanishes at some point, then applying the Hopf boundary lemma to the appropriate $\bar{u}_i$ at that point yields a contradiction to (5.4) or (5.5). Yet another proof of the positivity of ϕ_1 and ϕ_2, which does not use Lemma 5.1 at all, is the following. In Lemmas 4.2 and 4.3 of [3] it is shown that for $c > \sigma_0$, the operator $L + cD$ has an inverse $(L + cD)^{-1}$, defined on all of X_2, whose entries are convolution operators with positive kernels. Now, if α_1 and α_2 are positive and ϕ is not identically zero, then the entries of $\nabla N(\phi)$ are non-negative and not identically zero. Hence the function $(L + cD)^{-1}(\nabla N(\phi))$ is everywhere positive on $\mathbb{R}$. The desired result then follows immediately upon rewriting the solitary-wave equation (2.2) in the form $\phi = (L + cD)^{-1}(\nabla N(\phi))$.

Now for each $\mu \in \mathbb{R}$, define $w_i(x, y, \mu)$ for $(x, y) \in S_i$, $i = 1, 2, 3$, by

$$w_i(x, y, \mu) = \bar{u}_i(2\mu - x, y) - \bar{u}_i(x, y).$$

We will examine the behavior of w_i on the set

$$\Sigma_i(\mu) = \{(x, y) \in S_i : \ x \leq \mu\}.$$

Lemma 5.3. *Suppose α_1 and α_2 are positive. Then there exists $\eta_0 \in \mathbb{R}$ such that*

(i) for all $\mu \in \mathbb{R}$, if $w_2(x, y, \mu)$ attains a minimum value over $\Sigma_2(\mu)$ at some point (x_0, y_0) in $\Sigma_2(\mu)$, then either $x_0 > \eta_0$ or $w_2(x_0, y_0, \mu) \geq 0$.

(ii) for all $\mu \leq \eta_0$ and all $i \in \{1, 2, 3\}$, we have $w_i(x, y, \mu) \geq 0$ for all $(x, y) \in \Sigma_i(\mu)$.

Proof. Let functions $B_1 : \mathbb{R}^3 \to \mathbb{R}$ and $B_2 : \mathbb{R}^3 \to \mathbb{R}$ be defined by

$$B_1(p, q, r) = Q_1 p + \frac{\alpha_1}{2} p^2 + \gamma_1 q - \gamma_2 r$$

and

$$B_2(p, q, r) = Q_2 p + \frac{\alpha_2}{2} p^2 + \gamma_4 (1 + \theta_2 H_2) q - \gamma_3 (1 + \theta_2 H_2) r,$$

so that (5.4) and (5.5) take the form

$$B_1(\bar{u}_2(x, 0), \bar{u}_{1y}(x, 0), \bar{u}_{2y}(x, 0)) = 0 \qquad \text{for all } x \in \mathbb{R} \tag{5.6}$$

and

$$B_2(\bar{u}_2(x, -H_2), \bar{u}_{2y}(x, -H_2), \bar{u}_{3y}(x, -H_2)) = 0 \qquad \text{for all } x \in \mathbb{R}. \tag{5.7}$$

Since Q_1 and Q_2 are negative, and $\bar{u}_2 \to 0$ uniformly in y as $x \to -\infty$, we can find η_0 such that if $x \leq \eta_0$, then

$$\frac{\partial B_1}{\partial p} = Q_1 + \alpha_1 p < 0 \qquad \text{at } p = \bar{u}_2(x, 0) \tag{5.8}$$

and

$$\frac{\partial B_2}{\partial p} = Q_2 + \alpha_2 p < 0 \qquad \text{at } p = \bar{u}_2(x, -H_2). \tag{5.9}$$

We now prove (i) by contradiction. Suppose $x_0 \leq \eta_0$ and $w_2(x_0, y_0, \mu) < 0$. Since the functions w_i satisfy the same equation (5.3) on S_i as do the functions $\bar{u}_i$, and $w_i \to 0$ as $x \to -\infty$, we conclude from the maximum principle that (x_0, y_0) must lie on the boundary of $\Sigma_2(\mu)$. Moreover, since $w_2(\mu, y, \mu) = 0$ for all y, (x_0, y_0) must lie on the horizontal part of the boundary of $\Sigma_2(\mu)$, so that either $y_0 = 0$ or $y_0 = -H_2$.

Consider first the case $y_0 = 0$. Since $w_2(x, y, \mu)$ attains its minimum value on $\Sigma_2(\mu)$ at $(x_0, 0)$, we have $w_{2y}(x_0, 0, \mu) \leq 0$, and hence

$$\bar{u}_{2y}(2\mu - x_0, 0) \leq \bar{u}_{2y}(x_0, 0). \tag{5.10}$$

Also, since $w_1 = w_2$ for $y = 0$, the maximum principle implies that the minimum value of $w_1(x, y, \mu)$ on $\Sigma_1(\mu)$ is also attained at $(x_0, 0)$. Therefore $w_{1y}(x_0, 0, \mu) \geq 0$, and so

$$\bar{u}_{1y}(2\mu - x_0, 0) \geq \bar{u}_{1y}(x_0, 0). \tag{5.11}$$

On the other hand, since $w_2(x_0, 0, \mu) < 0$, we have

$$\bar{u}_2(2\mu - x_0, 0) < \bar{u}_2(x_0, 0). \tag{5.12}$$

Since $\alpha_1 > 0$, it follows from (5.8) and (5.12) that

$$\frac{\partial B_1}{\partial p} = Q_1 + \alpha_1 p < 0 \qquad \text{for all } p \in [\bar{u}_2(2\mu - x_0, 0), \bar{u}_2(x, 0)]. \tag{5.13}$$

Finally, combining (5.10), (5.11), (5.12) and (5.13), and recalling that γ_1 and γ_2 are positive, we obtain that

$$B_1(\bar{u}_2(2\mu - x_0, 0), \bar{u}_{1y}(2\mu - x_0, 0), \bar{u}_{2y}(2\mu - x_0, 0)) > B_1(\bar{u}_2(x_0, 0), \bar{u}_{1y}(x_0, 0), \bar{u}_{2y}(x_0, 0)),$$

contradicting (5.6).

We have shown that in the case $y_0 = 0$, we obtain a contradiction to (5.6). On the other hand, if $y_0 = -H_2$, an exactly similar argument leads to a contradiction of (5.7). Thus the proof of (i) is complete.

To prove (ii), suppose to the contrary that for some i and some $\mu \leq \eta_0$, $w_i(x, y, \mu)$ takes a negative value on $\Sigma_i(\mu)$. Then since $w_i \to 0$ as $x \to -\infty$, the maximum principle implies that $w_i(x, y, \mu)$ attains its minimum over $\Sigma_i(\mu)$ at some point on the boundary shared by $\Sigma_i(\mu)$ and $\Sigma_2(\mu)$. Hence w_2 takes a negative value on $\Sigma_2(\mu)$, and so attains a negative minimum value over $\Sigma_2(\mu)$ at some point (x_0, y_0) in $\Sigma_2(\mu)$. Since $x_0 \leq \mu \leq \eta_0$, this contradicts (i). $\square$

We can now state and prove the main theorem of this section.

Theorem 5.4. *Suppose that α_1 and α_2 are positive, and the u_i are not identically zero. Then there exists $\bar{\eta} \in \mathbb{R}$ such that*

(i) for all $i \in \{1,2,3\}$ and all $(x,y) \in S_i$,

$$u_i(x,y) = u_i(2\bar{\eta} - x, y).$$

(ii) for all $i \in \{1,2,3\}$ and all $(x,y) \in S_i$ such that $x < \bar{\eta}$ and $-(H_2 + H_3) < y < H_1$,

$$\frac{\partial u_i}{\partial x}(x,y) > 0.$$

Proof. Define the number $\bar{\eta}$ by

$$\bar{\eta} = \sup\left\{\eta : \text{ if } \mu \le \eta, \text{ then } w_i(x,y,\mu) \ge 0 \text{ for all } (x,y) \in \Sigma_i(\mu) \text{ and all } i \in \{1,2,3\}\right\}.$$

Lemma 5.4(ii) shows that $\bar{\eta} > -\infty$, and it follows easily from Lemma 5.1(ii) and Lemma 5.2 that $\bar{\eta} < \infty$.

To prove (i), it suffices to show that for each $i = 1,2,3$ we have $w_i(x,y,\bar{\eta}) = 0$ for all $(x,y) \in \Sigma_i(\bar{\eta})$.

By the definition of $\bar{\eta}$, we can find a sequence $\{\mu_k\}$ such that $\mu_k \to \bar{\eta}$ and for each k, there exists $i \in \{1,2,3\}$ for which $w_i(x,y,\mu_k)$ takes a negative minimum on $\Sigma_i(\mu_k)$. As noted in the proof of Lemma 5.3, it follows that $w_2(x,y,\mu_k)$ must take a negative minimum value on $\Sigma_2(\mu_k)$, and this value must be achieved at a point (x_k, y_k) where either $y_k = 0$ or $y_k = -H_2$. By passing to a subsequence, we may assume that either $y_k = 0$ for all k, or $y_k = -H_2$ for all k. We will consider the former of these two cases, the proof in the latter case being exactly similar.

From Lemma 5.3(i) it follows that $x_k > \eta_0$ for all k. Therefore the sequence $\{x_k\}$ is bounded, so by again passing to a subsequence we may assume that x_k converges to some number $\bar{x} \le \bar{\eta}$. Since $w_2(x_k, 0, \mu_k) < 0$ for all k, then $w_2(\bar{x}, 0, \bar{\eta}) \le 0$. Hence also $w_1(\bar{x}, 0, \bar{\eta}) \le 0$. On the other hand, from the definition of $\bar{\eta}$ it follows that $w_1(x,y,\bar{\eta}) \ge 0$ on $\Sigma_1(\bar{\eta})$ and $w_2(x,y,\bar{\eta}) \ge 0$ on $\Sigma_2(\bar{\eta})$. Therefore

$$w_1(\bar{x}, 0, \bar{\eta}) = w_2(\bar{x}, 0, \bar{\eta}) = 0, \tag{5.14}$$

and it follows that

$$w_{1y}(\bar{x}, 0, \bar{\eta}) \geq 0 \tag{5.15}$$

and

$$w_{2y}(\bar{x}, 0, \bar{\eta}) \leq 0. \tag{5.16}$$

We now consider separately the cases when $\bar{x} < \bar{\eta}$ and when $\bar{x} = \bar{\eta}$. Suppose first that $\bar{x} < \bar{\eta}$. Substitute into (5.4) the values $x = 2\bar{\eta} - \bar{x}$ and $x = \bar{x}$, and subtract the two resulting equations, using (5.14). There appears the identity

$$\gamma_1 w_{1y}(\bar{x}, 0, \bar{\eta}) = \gamma_2 w_{2y}(\bar{x}, 0, \bar{\eta}),$$

which together with (5.15) and (5.16) yields

$$w_{1y}(\bar{x}, 0, \bar{\eta}) = w_{2y}(\bar{x}, 0, \bar{\eta}) = 0.$$

Hence, applying the Hopf boundary lemma to $w_2(x, y, \bar{\eta})$ at the point $(\bar{x}, 0)$, we obtain that $w_2(x, y, \bar{\eta})$ is identically zero on $\Sigma_2(\bar{\eta})$. It then follows from the maximum principle that $w_1(x, y, \bar{\eta})$ is identically zero on $\Sigma_1(\bar{\eta})$ and $w_3(x, y, \bar{\eta})$ is identically zero on $\Sigma_3(\bar{\eta})$. Thus (i) has been proved in case $\bar{x} < \bar{\eta}$, and we may assume henceforth that $\bar{x} = \bar{\eta}$.

We proceed to investigate the derivatives of $w_2(x, y, \bar{\eta})$ at the point $(x, y) = (\bar{\eta}, 0)$. Notice first that since $w_2(x, y, \mu_k)$ attains a minimum over $\Sigma_2(\mu_k)$ at $(x_k, 0)$, and $x_k < \mu_k$, we must have $w_{2x}(x_k, 0, \mu_k) = 0$ for all k. Taking the limit as $k \to \infty$ then gives

$$w_{2x}(\bar{\eta}, 0, \bar{\eta}) = 0. \tag{5.17}$$

Since, for any μ and y,

$$w_{ix}(\mu, y, \mu) = -2\bar{u}_{ix}(\mu, y), \tag{5.18}$$

then (5.17) implies

$$\bar{u}_{2x}(\bar{\eta}, 0) = 0. \tag{5.19}$$

Also, clearly $w_2(\bar{\eta}, y, \bar{\eta}) = 0$ for $-H_2 \leq y \leq 0$, so

$$w_{2y}(\bar{\eta}, 0, \bar{\eta}) = w_{2yy}(\bar{\eta}, 0, \bar{\eta}) = 0. \tag{5.20}$$

Since w satisfies (5.3), then (5.20) implies

$$w_{2xx}(\bar{\eta}, 0, \bar{\eta}) = 0. \tag{5.21}$$

Finally we consider the mixed derivative $w_{2xy}(\bar{\eta}, 0, \bar{\eta})$. Observe that since the minimum value of $w_1(x, y, \bar{\eta})$ on $\Sigma_1(\bar{\eta})$ is attained at every point $(\bar{\eta}, y)$ in $\Sigma_1(\bar{\eta})$, then $w_{1x}(\bar{\eta}, y, \bar{\eta}) \leq 0$ for $0 \leq y \leq H_1$. Similarly, we have $w_{2x}(\bar{\eta}, y, \bar{\eta}) \leq 0$ for $-H_2 \leq y \leq 0$. Taking (5.17) into account, we conclude that $w_{1xy}(\bar{\eta}, 0, \bar{\eta}) \leq 0$ and $w_{2xy}(\bar{\eta}, 0, \bar{\eta}) \geq 0$, so that from (5.18) we obtain

$$\bar{u}_{1xy}(\bar{\eta}, 0) \geq 0 \tag{5.22}$$

and

$$\bar{u}_{2xy}(\bar{\eta}, 0) \leq 0. \tag{5.23}$$

On the other hand, differentiating (5.4) with respect to x and evaluating at $x = \bar{\eta}$ using (5.17), we obtain

$$\gamma_1 \bar{u}_{1xy}(\bar{\eta}, 0) = \gamma_2 \bar{u}_{2xy}(\bar{\eta}, 0),$$

which, together with (5.22) and (5.23), implies

$$\bar{u}_{1xy}(\bar{\eta}, 0) = \bar{u}_{2xy}(\bar{\eta}, 0) = 0.$$

In particular, it follows that

$$w_{2xy}(\bar{\eta}, 0, \bar{\eta}) = 0. \tag{5.24}$$

We have now shown (in (5.14), (5.17), (5.20), (5.21), and (5.24)) that, at the point $(x, y) = (\bar{\eta}, 0)$, $w_2(x, y, \bar{\eta})$ and all its partial derivatives of first and second order are equal to zero. It therefore follows from the Hopf corner-point lemma that $w_2(x, y, \bar{\eta})$ must be zero for all (x, y) in $\Sigma_2(\bar{\eta})$. As above, this is enough to conclude that (i) holds. Hence (i) has now been proved in all cases.

To prove (ii), we first note that if the u_i are not identically zero, then there does not exist any $\mu < \bar{\eta}$ such that $w_2(x, y, \mu)$ is identically zero on $\Sigma_2(\mu)$. For if there were such a μ, it would follow that the u_i are symmetric about μ as well as $\bar{\eta}$, and hence that the u_i are periodic of period $\mu - \bar{\eta}$, contradicting the fact that $u_i \to 0$ as $|x| \to \infty$.

Now observe that, if it were the case that $w_{2x}(\mu, 0, \mu) = 0$ for some $\mu < \bar{\eta}$, then the same chain as reasoning as above, starting with (5.17) and concluding with (5.24), would show that $w_2(x, y, \mu)$ and all its partial derivatives up to second order are zero at $(x, y) = (\mu, 0)$. The Hopf corner-point lemma would then imply that $w_2(x, y, \mu)$ is identically zero on $\Sigma_2(\mu)$, contradicting the result of the preceding paragraph. Therefore $w_{2x}(\mu, 0, \mu) \neq 0$. However, since $w_i(x, y, \mu) \geq 0$ on $\Sigma_i(\mu)$ and $w_i(\mu, y, \mu) = 0$, we must have

$$w_{ix}(\mu, y, \mu) \leq 0 \quad \text{for all } (\mu, y) \in \Sigma_i(\mu). \tag{5.25}$$

Hence $w_{2x}(\mu, 0, \mu) < 0$, so by (5.18) we have $\bar{u}_{2x}(\mu, 0) > 0$. It follows that $\bar{u}_{1x}(\mu, 0) > 0$ also. A similar argument shows that $\bar{u}_{2x}(\mu, -H_2) = \bar{u}_{3x}(\mu, -H_2) > 0$.

It remains to show that $\bar{u}_{ix}(\mu, y) > 0$ if $-(H_2 + H_3) < y < H_1$ and y is neither $-H_2$ nor 0. By (5.25), it suffices to show that $w_{ix}(\mu, y, \mu)$ cannot be zero for such y. But if indeed $w_{ix}(\mu, y, \mu) = 0$ for some i, then since (μ, y) is on the interior of the vertical boundary of $\Sigma_i(\mu)$, the Hopf boundary lemma implies that $w_i(x, y, \mu)$ is identically zero on $\Sigma_i(\mu)$. We know from above that i cannot equal 2, so either $i = 1$ or $i = 3$. But in either case, the fact that $w_i(x, y, \mu)$ is identically zero on $\Sigma_i(\mu)$ implies that $w_{2x}(x, y, \mu) = 0$ at one of the corner points of $\Sigma_2(\mu)$, and we are back to the situation of the preceding paragraph. Hence, in any case, we obtain a contradiction, and the proof of (ii) is complete. $\square$

Remark. If $\phi(x - ct)$ solves (4.2), then $-\phi(x - ct)$ solves (4.2) with α_1 and α_2 replaced by $-\alpha_1$ and $-\alpha_2$. Hence it follows from Theorem 5.4 that if α_1 and α_2 are negative, then the conclusions of the theorem hold for $-\phi$. We do not yet know an analogue of Theorem 5.4 in the case when α_1 and α_2 have different signs.

It is possible to extend Theorem 5.4 to cases in which the u_i do not tend to zero in both horizontal directions.

Theorem 5.5. *Suppose that α_1 and α_2 are positive. Suppose also that functions $u_i \in C^2(S_i)$ are given which satisfy all the conditions of Lemma 5.1, except that (ii) is replaced by the requirements that, for some $\beta > 0$ and all $i \in \{1, 2, 3\}$, we have*

$$\lim_{x \to -\infty} u_i = 0 \quad \textit{and} \quad \lim_{x \to +\infty} u_i = \beta,$$

both limits being uniform in y. Then for all $i \in \{1,2,3\}$ and all $(x,y) \in S_i$ such that $-(H_2+H_3) < y < H_1$, we have

$$\frac{\partial u_i}{\partial x}(x,y) > 0.$$

Proof. Define $\bar{u}_i$ and w_i as before. Under the given assumptions on u_i, the proof of Lemma 5.2 still goes through, showing that $\bar{u}_i \geq 0$ on S_i for $i = 1,2,3$. Also, although it is now no longer necessarily the case that $w_i \to 0$ as $x \to -\infty$, it is still true that

$$\liminf_{x \to -\infty} w_i(x,y,\mu) \geq 0,$$

and it may be easily checked that this is sufficient for the proof of Lemma 5.3 to be carried out as before. Thus Lemma 5.3 still holds, and in particular we may define $\bar{\eta}$ as before with the assurance that $\bar{\eta} > -\infty$.

Now if $\bar{\eta} < \infty$, then the proof of Theorem 5.4(i) shows that the u_i are symmetric about $\bar{\eta}$, which contradicts our assumptions about the behavior of u_i as $x \to \pm\infty$. Therefore we must have $\bar{\eta} = \infty$. The desired result is now obtained by noticing that the proof of Theorem 5.4(ii) goes through unchanged in the present situation. □

Remark. The functions u_i described in Theorem 5.5 do not arise from L^2 solitary-wave solutions of (4.2): indeed, since $\phi_1(x) = u_1(x,0)$ and $\phi_2(x) = u_2(x,-H_2)$ are not in L^2, the operator L will not in general be well-defined at $\phi = (\phi_1, \phi_2)$. On the other hand, if one were to derive equations modeling bore-like waves at the interfaces of a three-layered fluid by a procedure analogous to the one used to derive (4.2) for localized waves, then ϕ would represent a valid solution to such a system. Thus Theorem 5.5 can be interpreted as a result for a system of equations modelling internal bores.

Finally we note that the above arguments also yield a symmetry result for solitary-wave solutions of a scalar equation derived by Kubota, Ko and Dobbs [24] as a model for long waves in a stratified fluid at the interface between two layers of constant density, one layer having (non-dimensionalized) depth equal to H_1 and the other layer having depth H_2. After a suitable choice of variables, the Kubota-Ko-Dobbs equation may be put in the form

$$h_t + hh_x - \beta_1(M_1 h)_x - \beta_2(M_2 h)_x = 0, \tag{5.26}$$

where $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ and β_1, β_2 are positive real numbers. The operators M_1 and M_2 are defined as in Section 4b, and in place of (4.3) we now have $\sigma_0 = 0$. In case $H_1 = H_2$, (5.26) is known as the Intermediate Long Wave (ILW) equation, and has been extensively studied in the literature devoted to completely integrable equations (cf. [1]). The ILW equation has, for each $c > 0$, a solitary-wave solution $\phi(x - ct)$ given by an explicit formula in terms of exponential functions; in particular this formula shows that ϕ is symmetric about the point where it attains its maximum value and is strictly decreasing away from that point. Further, it is known [5] that ϕ is, up to translation, the unique solitary wave solution of ILW with wavespeed c. In case $H_1 \neq H_2$, it is known (see Theorem 3.1 of [3]) that for each $c > 0$, (5.26) has at least one solitary-wave solution $\phi(x - ct)$ which is symmetric about its maximum, but it remains an open question whether ϕ is unique up to translation. The following result is therefore of interest.

Theorem 5.6. *Let $\phi \in L^2(\mathbb{R})$ be such that $\phi(x - ct)$ solves (5.26) for some $c > 0$. Then there exists $\bar{\eta} \in \mathbb{R}$ such that $\phi(2\bar{\eta} - x) = \phi(x)$ for all $x \in \mathbb{R}$ and $\phi'(x) > 0$ for all $x < \bar{\eta}$.*

To prove Theorem 5.6, one first observes that solitary-wave solutions of (5.26) are associated with the same kind of elliptic boundary-value problem as specified in Lemma 5.1, except that now only two strips are involved. More precisely, let S_1 and S_2 be the infinite strips in $\mathbb{R}^2$ defined by

$$S_1 = \mathbb{R} \times [0, H_1],$$
$$S_2 = \mathbb{R} \times [-H_2, 0].$$

Then for ϕ as in the statement of Theorem 5.6, one finds that there exist functions $u_1 \in C^\infty(S_1)$ and $u_2 \in C^\infty(S_2)$ such that

(i) $\Delta u_1 = 0$ on S_1, and $\Delta u_2 = 0$ on S_2,

(ii) u_1 and u_2 tend to 0 uniformly in y as $|x| \to \infty$,

(iii) $u_1 = 0$ for $y = H_1$,

(iv) $u_2 = 0$ for $y = -H_2$,

(v) $u_1 = u_2 = \phi$ for $y = 0$, and

(vi) $\left[-c + \frac{\beta_1}{H_1} + \frac{\beta_2}{H_2}\right] \phi + \frac{1}{2}\phi^2 + \beta_1 u_{1y} - \beta_2 u_{2y} = 0$ for $y = 0$.

Now choose θ_1 and θ_2 such that $1+\theta_1 H_1$ and $1+\theta_2 H_2$ are positive numbers, and

$$Q = -c + \frac{\beta_1}{H_1}(1+\theta_1 H_1) + \frac{\beta_2}{H_2}(1+\theta_2 H_2) < 0.$$

Defining g_1, g_2 and $\bar{u}_1, \bar{u}_2$ by the same formulas as given above prior to Lemma 5.2, we find that (vi) implies

$$Q\bar{u}_2 + \frac{1}{2}\bar{u}_2^2 + \beta_1 \bar{u}_{1y} - \beta_2 \bar{u}_{2y} = 0 \qquad \text{for } y = 0.$$

From here the proof of Theorem 5.6 proceeds exactly like the proof of Theorem 5.4, and we can safely omit the details.

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