Bound-state solutions and well-posedness of the dispersion-managed nonlinear Schrödinger and related equations

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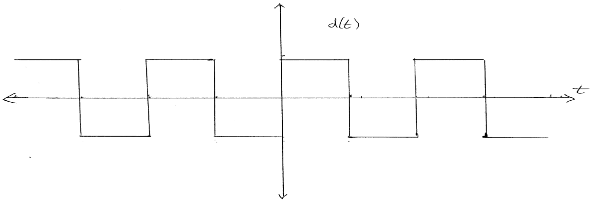
The one-dimensional nonlinear Schrödinger (NLS) equation

$$iu_t + u_{xx} + |u|^2 u = 0$$

models the propagation of the envelopes of oscillatory pulses in a homogenous medium. In many applications, a more suitable model equation is

$$iu_t + d(t)u_{xx} + |u|^2 u = 0.$$

For example, in dispersion-managed optical fibers, one would take $d(t)$ to be a periodic function of the spatial variable t.



For the NLS equation with periodic coefficient $d(t)$, Antonelli, Saut, & Sparber (2012) proved global well-posedness of the initial-value problem in $L^2(\mathbb{R})$, assuming $d(t)$ is piecewise constant.

The NLS equation is known to have *bound-state* solutions of the form $u(x,t) = e^{i\omega t}\phi(x)$ where $\omega \in \mathbb{R}$ is arbitrary and $\phi(x)$ is a localized (hyperbolic secant) function. Numerical studies suggest that the periodic-coefficient equation has solutions which are similar in that they are localized in x and periodic in t. (See, e.g., Bronski & Kutz 1997; or Grigoryan et al. 1997.)

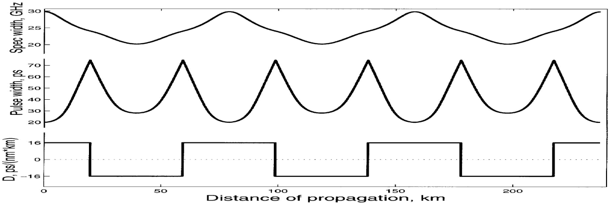


Figure: Numerical simulation of solitons in dispersion managed fibers; due to A. Berntson at Chalmers U., Sweden.

DMNLS equation

For the equation

$$iu_t + d(t)u_{xx} + |u|^2 u = 0,$$

if $d(t) = \frac{1}{\epsilon}\delta(\frac{t}{\epsilon}) + \alpha$, where $\delta(t)$ is periodic and has mean value zero, then in the limit as $\epsilon \to 0$, the solution with given initial data $u(x, 0)$ is well approximated in rescaled variables by the solution of the *dispersion-managed NLS equation* (Gabitov & Turitsyn 1996, Zharnitsky et al., 2001):

$$iu_t + \alpha u_{xx} + \int_0^1 T(s)^{-1} \left[|T(s)u|^2 T(s)u \right] \, ds = 0$$

with the same initial data. Here $T(t) = e^{-i(\int_0^t \delta(t')dt')\partial_x^2}$ is the solution operator for the initial-value problem for the linear Schrödinger equation

$$iu_t + \delta(t)u_{xx} = 0.$$

In fact, the DMNLS equation can be written in Hamiltonian form $u_t = -i\nabla E(u)$, with Hamiltonian

$$E(u) = \frac{\alpha}{2}\int_{-\infty}^{\infty} |u_x|^2 \ dx - \frac{1}{4}\int_0^1 \int_{-\infty}^{\infty} |T(s)u|^4 \ dx \ ds.$$

Thus the *energy space* is $H^1(\mathbb{R})$ in case $\alpha \neq 0$, but is $L^2(\mathbb{R})$ in case $\alpha = 0$.

Note that the value $\alpha = 0$ is consistent with the assumptions underlying the derivation of DMNLS as a model equation, and is within the range of values of α one would expect to see in applications.

We assume throughout this talk that $\delta(t)$ is piecewise C^1 and bounded away from zero.

We can ask:

• Does DMNLS have bound-state solutions like those of NLS, of the form $u(x,t) = e^{i\omega t}\phi(x)$ for localized ϕ?

• If so, what can we say about the stability of these solutions? In particular, is DMNLS well-posed in energy space? (In case $\alpha > 0$, energy space is H^1, so well-posedness is easy.)

Bound-state solutions $u(x,t) = e^{i\omega t}\phi(x)$ correspond to critical points ϕ for the variational problem

$$\inf \left\{ E(f) : f \in X, \|f\|_{L^2(\mathbb{R}^2)} = \lambda \right\},$$

where X is energy space (H^1 if $\alpha \neq 0$, L^2 if α=0.) Minimizers of this problem are called *ground states*.

In case $\alpha > 0$, Zharnitsky et al. (2001) proved that the set S of minimizers is nonempty, and in fact every minimizing sequence f_j has a subsequence which converges in H^1 to S. It follows that S is a stable set for the initial-value problem for DMNLS in H^1.

In case $\alpha < 0$, Zharnitsky et al. showed bound states (if they exist) cannot be minimizers, and did numerical experiments suggesting that either bound states do not exist or are not stable.

CR equation

Faou, Germain, & Hani (2016) considered solutions of the cubic nonlinear Schrodinger equation in $\mathbb{R}^2$,

$$iu_t + \Delta u + |u|^2 u = 0,$$

which have small amplitude ϵ and are periodic with large period L in both spatial variables. They showed that as $\epsilon \to 0$ and $L \to \infty$ with $\epsilon L^2 \sim 1$, solutions are well approximated by those of the *continuous resonant* (CR) equation,

$$iu_t + \int_{-\infty}^{\infty} U_2(s)^{-1} \left[|U_2(s)u|^2 U_2(s)u \right] \, ds = 0,$$

in which $U_2(t) = e^{-it\Delta}$ is the solution operator for the linear Schrödinger equation

$$iu_t + \Delta u = 0 \quad \text{on } \mathbb{R}^2.$$

This equation is of Hamiltonian form $u_t = -i\nabla E_2(u)$, with Hamiltonian

$$E_2(u) = -\frac{1}{4}\int_{-\infty}^{\infty}\int_{\mathbb{R}^2} |U_2(s)u|^4 \, dx \, ds.$$

Bound-state solutions $u(x,t) = e^{i\omega t}\phi(x)$ correspond to critical points ϕ for the variational problem

$$\inf\left\{E_2(f) : \|f\|_{L^2(\mathbb{R}^2)} = \lambda\right\}.$$

Minimizers are the functions ϕ which attain the best constant S_2 in the Strichartz inequality

$$\left(\int_{-\infty}^{\infty}\int_{\mathbb{R}^2} |U_2(s)f|^4 \, dx \, ds\right)^{1/4} \le S_2 \left(\int_{\mathbb{R}^2} |f|^2 \, dx\right)^{1/2}.$$

Thus ground states of CR correspond to *maximizers* for the Strichartz inequality.

In fact, it is known that ϕ is a maximizer for the Strichartz inequality if and only if $\phi(x) = \alpha e^{-\beta|x-a|^2 + b\cdot x}$, where $\alpha \neq 0$, $\beta > 0$, $a \in \mathbb{R}^2$, and $b \in \mathbb{R}^2$ are arbitrary. Hence $S_2 = (1/2)^{1/2}$ (Foschi, 2007).

Hundertmark and Zharnitsky (2006) gave a beautiful proof of this fact based on the following geometric interpretation of $E_2(f)$:

$$E_2(f) = -\frac{1}{16}\langle f \otimes f, P(f \otimes f)\rangle_{L^2(\mathbb{R}^4)},$$

where $(f \otimes f)(x_1, x_2, x_3, x_4) := f(x_1, x_2)f(x_3, x_4)$, and P is the orthogonal projection of $L^2(\mathbb{R}^4)$ onto the subspace of functions which are invariant under all rotations of $\mathbb{R}^4$ which fix both the points $(1, 0, 1, 0)$ and $(0, 1, 0, 1)$.

A curious fact is that $E_2(f) = E_2(\hat{f})$ for all $f \in L^2(\mathbb{R}^2)$. In fact, u is a solution of the CR equation if and only if $\hat{u}$ is a solution.

1DCR equation

A one-dimensional analogue of the CR equation is

$$iu_t + \int_{-\infty}^{\infty} U_1(s)^{-1} \left[|U_1(s)u|^4 U_1(s)u \right] \; ds = 0,$$

where $U_1(t) = e^{-it\partial_x^2}$ is the solution operator for the linear Schrödinger equation

$$iu_t + u_{xx} = 0 \quad \text{on } \mathbb{R}.$$

The 1DCR equation would be expected to model the behavior of small-amplitude solutions with large period L for the one-dimensional quintic NLS equation

$$iu_t + u_{xx} + |u|^5 u_x = 0.$$

Here the Hamiltonian is

$$E_1(u) = -\frac{1}{6}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |U_2(s)u|^6 \; dx \; ds.$$

Bound-state solutions correspond to critical points ϕ for

$$\inf\left\{E_1(f) : \|f\|_{L^2(\mathbb{R}^2)} = \lambda\right\},$$

and ground states are maximizers for the Strichartz inequality

$$\left(\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |U_1(s)f|^6 \; dx \; ds\right)^{1/6} \leq S_1\left(\int_{\mathbb{R}^2} |f|^2 \; dx\right)^{1/2}.$$

Foschi showed that ϕ is a maximizer iff it is a Gaussian, and $S_1 = (1/12)^{1/12}$. Hundertmark and Zharnitsky also gave a geometric interpretation of E_1.

Global well-posedness for DMNLS and 1DCR

Theorem

Suppose $r \geq 0$.

- *For every* $u_0 \in H^r$ *and every* $M > 0$, *the DMNLS equation (for any* $\alpha \in \mathbb{R}$*) and the 1DCR equation have unique strong solutions* $u \in C([0, M]; H^r)$ *with initial data* u_0. *We have* $u \in L_t^q([0, M], L_x^p(\mathbb{R}))$ *for every* p *and* q *satisfying* $2 \leq p \leq \infty$, $4 \leq q \leq \infty$, *and* $2/q = (1/2) - (1/p)$.
- *The map taking* u_0 *to* u *is a locally Lipschitz map from* H^r *to* $C([0, M]; H^r)$.

Remark: the same result holds for $iu_t + u_{xx} + |u|^2 u = 0$, and is known to be sharp in the sense that it does not hold when $r < 0$.

Theorem (Kunze, 2004)

Suppose $\alpha = 0$. Then for each $\lambda > 0$, the variational problem

$$I_\lambda := \inf \{E(f) : \|f\|_{L^2} = \lambda\} \quad (1)$$

has a non-empty set of minimizers S. Moreover, every minimizing sequence f_j has a subsequence which converges in L^2 to S. It follows that S is a stable set for the initial-value problem in L^2.

Remark: this is not yet an orbital stability result, since we do not yet know the structure of S.

Kunze's idea (see also Kunze, Moeser, & Zharnitsky [2005]) is to apply the concentration compactness lemma both to f_j and to the Fourier transforms $\hat{f}_j$, ruling out "vanishing" and "splitting" by using the subadditivity and negativity of I_λ as a function of λ. We conclude that both f_j and $\hat{f}_j$, when suitably translated, are "tight".

Now, since $\hat{f}_j$ is tight, we can decompose f_j into a low-frequency part f_j^L which is bounded in H^1, uniformly in j, and a high-frequency part which is small in L^2, uniformly in j. But since f_j is tight, then so is f_j^L. Using the compactness of the embedding of H^1 into L^2 on bounded domains, we can then conclude that f_j^L has a subsequence which converges strongly in L^2. Hence so also does f_j.

The global well-posedness result for 1DCR is sharp in the sense that it does not hold for $r < 0$.

Theorem

Suppose $r < 0$ and $M > 0$. There exists $B > 0$ and $C > 0$ such that for every $\delta > 0$, there exist two solutions $u(x,t)$ and $v(x,t)$ of 1DCR in $C([0,M],H^r)$, with initial data u_0 and v_0, for which $\|u_0\|_{H^r} \le B$, $\|v_0\|_{H^r} \le B$,

$$\|u_0 - v_0\|_{H^r} < \delta, \tag{2}$$

and

$$\|u(x,M) - v(x,M)\|_{H^r} \ge C. \tag{3}$$

This shows that that there cannot exist a locally uniformly continuous map from initial data to solutions in H^r when $r < 0$.

This is proved by taking

$$u_0(x) = \beta\omega_1 e^{iNx}\phi(\omega_1 x), \quad v_0(x) = \beta\omega_2 e^{iNx}\phi(\omega_2 x),$$

where $\phi(x)$ is a (Gaussian) bound-state solution, and

$$\beta = N^{-r-(1/4)},$$
$$\omega_1 = \sqrt{N},$$
$$\omega_2 = \sqrt{N}(1+\delta),$$

and $N > 0$ is a suitable large number.

Remark: The proof depends on knowing explicitly how the solution behaves when the initial data is dilated:

$$u(x,t) = \beta\omega_1 e^{i\beta^4\omega_1^2 t} e^{iNx}\phi(\omega_1 x)$$
$$v(x,t) = \beta\omega_2 e^{i\beta^4\omega_2^2 t} e^{iNx}\phi(\omega_2 x).$$

Such formulas are, however, not available for the DMNLS equation.