The 65th William Lowell Putnam Mathematical Competition
Saturday, December 4, 2004

A–1 Basketball star Shanille O'Keal's team statistician keeps track of the number, $S(N)$, of successful free throws she has made in her first N attempts of the season. Early in the season, $S(N)$ was less than 80% of N, but by the end of the season, $S(N)$ was more than 80% of N. Was there necessarily a moment in between when $S(N)$ was exactly 80% of N?

A–2 For $i = 1,2$ let T_i be a triangle with side lengths a_i, b_i, c_i, and area A_i. Suppose that $a_1 \le a_2, b_1 \le b_2, c_1 \le c_2$, and that T_2 is an acute triangle. Does it follow that $A_1 \le A_2$?

A–3 Define a sequence $\{u_n\}_{n=0}^{\infty}$ by $u_0 = u_1 = u_2 = 1$, and thereafter by the condition that

$$\det \begin{pmatrix} u_n & u_{n+1} \\ u_{n+2} & u_{n+3} \end{pmatrix} = n!$$

for all $n \ge 0$. Show that u_n is an integer for all n. (By convention, $0! = 1$.)

A–4 Show that for any positive integer n, there is an integer N such that the product $x_1 x_2 \cdots x_n$ can be expressed identically in the form

$$x_1 x_2 \cdots x_n = \sum_{i=1}^{N} c_i (a_{i1} x_1 + a_{i2} x_2 + \cdots + a_{in} x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers $-1, 0, 1$.

A–5 An $m \times n$ checkerboard is colored randomly: each square is independently assigned red or black with probability $1/2$. We say that two squares, p and q, are in the same connected monochromatic component if there is a sequence of squares, all of the same color, starting at p and ending at q, in which successive squares in the sequence share a common side. Show that the expected number of connected monochromatic regions is greater than $mn/8$.

A–6 Suppose that $f(x,y)$ is a continuous real-valued function on the unit square $0 \le x \le 1, 0 \le y \le 1$. Show that

$$\int_0^1 \left(\int_0^1 f(x,y)\,dx \right)^2 dy + \int_0^1 \left(\int_0^1 f(x,y)\,dy \right)^2 dx$$
$$\le \left(\int_0^1 \int_0^1 f(x,y)\,dx\,dy \right)^2 + \int_0^1 \int_0^1 (f(x,y))^2\,dx\,dy.$$

B–1 Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$c_n r,\ c_n r^2 + c_{n-1} r,\ c_n r^3 + c_{n-1} r^2 + c_{n-2} r,$$
$$\ldots,\ c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r$$

are integers.

B–2 Let m and n be positive integers. Show that

$$\frac{(m+n)!}{(m+n)^{m+n}} < \frac{m!}{m^m} \frac{n!}{n^n}.$$

B–3 Determine all real numbers $a > 0$ for which there exists a nonnegative continuous function $f(x)$ defined on $[0,a]$ with the property that the region

$$R = \{(x,y); 0 \le x \le a, 0 \le y \le f(x)\}$$

has perimeter k units and area k square units for some real number k.

B–4 Let n be a positive integer, $n \ge 2$, and put $\theta = 2\pi/n$. Define points $P_k = (k,0)$ in the xy-plane, for $k = 1,2,\ldots,n$. Let R_k be the map that rotates the plane counterclockwise by the angle θ about the point P_k. Let R denote the map obtained by applying, in order, R_1, then $R_2, \ldots,$ then R_n. For an arbitrary point (x,y), find, and simplify, the coordinates of $R(x,y)$.

B–5 Evaluate

$$\lim_{x \to 1^-} \prod_{n=0}^{\infty} \left(\frac{1 + x^{n+1}}{1 + x^n} \right)^{x^n}.$$

B–6 Let $\mathscr{A}$ be a non-empty set of positive integers, and let $N(x)$ denote the number of elements of $\mathscr{A}$ not exceeding x. Let $\mathscr{B}$ denote the set of positive integers b that can be written in the form $b = a - a'$ with $a \in \mathscr{A}$ and $a' \in \mathscr{A}$. Let $b_1 < b_2 < \cdots$ be the members of $\mathscr{B}$, listed in increasing order. Show that if the sequence $b_{i+1} - b_i$ is unbounded, then

$$\lim_{x \to \infty} N(x)/x = 0.$$