The Classical Inequalities

Putnam Practice

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Arithmetic mean - Geometric mean inequality says that for any n non-negative real numbers $a_1, ...a_n$ we have:

$$\frac{a_1 + a_2 + ... + a_n}{n} \geq \sqrt[n]{a_1 a_2 ... a_n}$$

and equality holds $\Leftrightarrow a_1 = a_2 = ... = a_n$. Define the **Harmonic mean** by

$$H = \frac{n}{1/a_1 + 1/a_2 + ... + 1/a_n}$$

then it is not hard to check that

$$HM \leq GM \leq AM.$$

Theorem 1 (Power mean inequality) *Let $a_1, ...a_n$ be positive real numbers, and let α be real. Let*

$$M_\alpha(a_1, ...a_n) = (\frac{a_1^\alpha + ... + a_n^\alpha}{n})^{1/\alpha}, \alpha \neq 0$$

and

$$M_0 = \sqrt[n]{a_1 ... a_n}.$$

Then M_α is an increasing function of α unless $a_1 = a_2 = ... = a_n$ (in which case M_α is constant).

Theorem 2 (Cauchy inequality) *For arbitrary real numbers $a_1, a_2, ...a_n$ and $b_1, b_2, ...b_n$ we have*

$$(a_1 b_1 + ... + a_n b_n)^2 \leq (a_1^2 + ... + a_n^2)(b_1^2 + ... + b_n^2)$$

Furthermore, equality holds if and only if there are two numbers λ, μ not both zero, such that $\lambda a_i = \mu b_i$ for all i.

Let $x = (x_1, ...x_n) \in \mathbb{R}^n$ and norm of x be

$$\|x\| = (x_1^2 + ... + x_n^2)^{1/2}.$$

Theorem 3 *For any two elements $x, y \in \mathbb{R}^n$*

$$\|x + y\| \leq \|x\| + \|y\|.$$

Equality holds if and only if there exist λ, μ not both zero such that $\lambda x = \mu y$.

Problems:

1. Show that if a, b, c are non-negative numbers, then

$$(a + b)(b + c)(c + a) \geq 8abc$$

2. (1975 Putnam) Let $H_n = 1 + 1/2 + ... + 1/n$. Show that

$$n(n + 1)^{1/n} < n + H_n$$

for every $n > 1$

3. Show that if a, b, c are positive real numbers, then

$$\frac{a^2}{b^2} + \frac{b^2}{c^2} + \frac{c^2}{a^2} \geq \frac{b}{a} + \frac{c}{b} + \frac{a}{c}$$

4. Show that if $a, b > 0$ and $a + b = 1$, then

$$(a + 1/a)^2 + (b + 1/b)^2 \geq 25/2$$

5. Let a, b, c be positive real numbers. Prove that

$$1/a + 1/b + 1/c \leq \frac{a^8 + b^8 + c^8}{a^3 b^3 c^3}$$

6. Prove Cauchy Inequality.

Hint: 1) $b_i = 0$, true

2) $b_1^2 + ...b_n^2 > 0$. Consider the polynomial $P(x) = (a_1 - b_1 x)^2 + (a_2 - b_2 x)^2 + ... + (a_n - b_n x)^2$. Note $P(x) \geq 0$ for any x, in particular for

$$x = \frac{a_1 b_1 + ... + a_n b_n}{b_1^2 + ... + b_n^2}$$